The Algebraic-Topological Basis For Network Analogies and the Vector Calculus

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Network analogies have been devised for a wide variety of physical systems, including many that are describable by the vector calculus. An explanation for this remarkable versatility of the network concept may be given in terms of the algebraic-topological principles upon which both network theory and the vector calculus are founded.

The network problem, though usually regarded as an electrical phenomenon, may be treated as a mathematical abstraction having two principal ingredients: (a) a topological structure, called a linear graph; and (b) an associated algebraic structure. Once the properties of these two ingredients are identified, ground rules for setting up network analogies become easy to establish. The interpretation of these ground rules in the case of electrical, mechanical, and structural systems is discussed.

By extending this abstract treatment of the linear graph to higher-dimensional topological structures, specifically those containing surface and volume elements as well as points and lines, a direct link-up between network theory and the vector calculus becomes possible. Two particularly interesting results of this development which will be described are:

1. A novel topological interpretation of Maxwell's equations for the electromagnetic field;
2. The validation of network representations for two large classes of partial differential equation.

INTRODUCTION

The wide variety of physical systems for which network analogies have been devised [1]-[14] hints strongly that there is something fundamental in the network concept. Indeed, the fact that a significant portion of physics and engineering lies within the scope of network theory is becoming more widely recognized, and a network approach to a broad spectrum of engineering systems has already been adopted by some authors [15], [16].

A satisfying explanation for the remarkable versatility of the network concept can be obtained by treating the network problem as an abstraction so that its underlying mathematical character may be exposed [17]. Once this has been done, firm ground rules for setting up network models become easy to establish. Indeed, the task of applying the mathematics of the network problem to different physical situations becomes almost a formality.

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In essence, network analysis is a very practical application of algebraic topology [18]–[23], a fact of which few mathematicians, except Roth, seem to be aware. In a fundamental paper [18], Roth showed that Kirchhoff’s current law is the electrical counterpart of what the topologist calls a homology sequence of a linear graph, while Kirchhoff’s voltage law corresponds to a cohomology sequence. Both of these sequences have long played a central role in the theory of topological complexes [22], [23]. But Roth showed that an additional transformation, not previously used by topologists, was needed to complete the topological description of the network problem. This transformation (technically an isomorphism, or 1-to-1 correspondence) interrelates the homology and cohomology sequences and corresponds electrically to Ohm’s law.

Roth’s characterization of the network problem not only has an inherent elegance that should appeal strongly to mathematicians, but also provides a deep insight into the basic nature of the network concept in a way that readily accounts for the ubiquity of this concept. What is more, as we shall show by extending this treatment to topological complexes of higher dimension than the linear graph, a direct link-up with the vector calculus becomes possible. Thus, the existence of higher-dimensional network analogies for systems usually described by the vector calculus comes to light. Two particularly interesting results of this new development are:

1. A novel topological interpretation of Maxwell’s equations for the electromagnetic field;

2. The validation of network representations for two large classes of partial differential equation.

There are several significant features of the evidently intimate relation between algebraic topology, network theory, and the vector calculus which are worth noting. First, algebraic topology deals with very simple but fundamental properties of the space in which physical phenomena happen. Second, network theory is founded directly on the most elemental principles of algebraic topology. Third, the very same principles inevitably come into play in the derivation of the vector calculus, which starts by considering certain numerical quantities associated with what amounts to a network of discrete points, lines, surface and volume elements interconnected with each other.

The algebraic relations between the quantities associated with the elements of this network define, in effect, a network problem of sorts. Even though the elements of this network are then allowed to shrink to infinitesimal size, the essentially topological character of the relations between the variables involved remains unchanged. As a consequence, it turns out that the familiar operations of gradient, curl, and divergence are counterparts, for the continuum, of standard topological operations in a discrete network. Accordingly, it is not surprising to find network models being used as an alternative to the vector calculus in numerical analytic treatments of many physical phenomena.

In this paper, which is a revision and extension of an earlier paper on the basis for network analogies [17], we shall discuss the algebraic topological basis for both network analysis and the vector calculus. First, we shall describe an abstract formulation of the network problem for a linear graph based on Roth’s
work [18]-[20]. This problem involves two principal ingredients: (a) the properties of a topological structure, namely the linear graph; and (b) the characteristics of an associated algebraic structure.

After describing these two ingredients, we will formally state two variations of the network problem and outline their solutions. We will then interpret the network problem in terms of its electrical, mechanical, and structural embodiments. Next, we will show how the network problem may be extended to a higher-dimensional topological complex involving not only points and lines, but also surface and volume elements, all interconnected.

Using the algebraic topological properties of this 3-complex and a dual 3-complex, we will develop a correspondence with the operational structure of the vector calculus. On this basis, we will describe a novel interpretation of the equations of the electrostatic and magnetostatic fields which can then be used to justify the existence of network models for two important classes of partial differential equation.

Finally, by extending this interpretation to the case of electrodynamical and magnetodynamical fields, we arrive at an illuminating explanation of how these fields are coupled together according to Maxwell’s equations.

**TOPOLOGICAL PROPERTIES OF LINEAR GRAPHS**

In this section we review the principal topological properties of linear graphs [24]. More detailed treatments of graph theory may be found in the literature [25]-[27]. We assume that such terms as oriented graph, node, branch, mesh, tree, link (or chord), etc., are familiar to the reader, and we will designate the number of branches, nodes, meshes, node-pairs, and separate parts (or subgraphs) by the letters \( b, n, m, p, \) and \( k \), respectively. The Euler-Poincaré relation then becomes \( b = m + p \), where \( p = n - \chi \). In the following discussion, however, we consider only the case of a connected graph, \( (k = 1) \).

As is well known, the connectivity relations of any oriented linear graph may be completely specified by means of what may be called the augmented branch-node incidence matrix, designated \( \bar{A} \), whose elements are \( +1, -1, \) and \( 0 \). If the rows of \( \bar{A} \) correspond to branches and the columns to nodes, the \( ij \)th element of \( \bar{A} \) is defined as follows:

\[
\bar{a}_{ij} = (+1, -1, 0) \quad \text{if the } i\text{th branch is (positively, negatively, not) incident on the } j\text{th node.}
\]

Since each row of \( \bar{A} \) contains both a single \( +1 \) and a single \( -1 \), its columns are linearly dependent. Deleting from \( \bar{A} \) the column corresponding to the datum or reference node of the graph, we obtain what may be called simply the branch node-pair matrix, designated \( A \), in which the columns are now linearly independent.

If a tree is chosen in the graph and the rows of \( A \) then properly arranged, \( A \) may be partitioned into the submatrix \( A_T \), referring to the tree branches only, and the submatrix \( A_L \), referring to the links, or tree-complement. A new topological matrix, called the node-to-datum-path matrix and designated \( B_T \), may now be de-
fined which also refers to the tree and which is related to $A_T$ in an interesting way. Again letting rows correspond to branches and columns to nodes, the $ij$th element of $B_T$ is defined as follows:

$$b_{ij} = (+1, -1, 0)$$ if the $i$th branch is (positively, negatively, not) included in the $j$th node-to-datum path.

Since the tree contains $(n - 1)$ branches and $(n - 1)$ nondatum nodes, both $A_T$ and $B_T$ are square matrices. Also, both matrices are invertible, and it turns out that for any tree the matrix $B_T$, when transposed, is equal to the inverse of $A_T$. That is,

$$A_T^{-1} = B_T^t$$

(1)

For a proof of this relation, see reference [24].

When a link, that is, any branch in the tree complement, is added to the tree, it forms a unique closed path called a basic mesh. Since every mesh in a graph can be described as a linear combination of basic meshes, and since the basic meshes can be unambiguously determined once the tree has been designated, it is particularly convenient to use these meshes as a basis for any mesh analysis. Furthermore, since each basic mesh contains only one link, considerable simplification is achieved by choosing the mesh orientation to coincide with that of its identifying link.

The connectivity relations of the branches and meshes of a graph may be specified by means of the branch-mesh matrix, designated $C$. If the rows of $C$ correspond to branches and columns to meshes (or to their identifying links), the $ij$th element of $C$ is defined as follows:

$$c_{ij} = (+1, -1, 0)$$ if the $i$th branch is (positively, negatively, not) included in the $j$th basic mesh.

With the rows of $C$ arranged in similar fashion to those of $A$, $C$ may be partitioned into submatrices $C_T$ and $C_L$. Then, since each basic mesh includes its identifying link in a positive sense, by convention, it follows that the submatrix $C_L$, which refers to the links, is a unit matrix, $U_L$. Thus, the interesting part of the branch-mesh matrix is the submatrix $C_T$ which refers to the tree branches only.

One of the most fundamental properties of a linear graph is the fact that the branch-node and branch-mesh matrices obey the relations

$$A'C = 0 \quad \text{and/or} \quad C'A = 0$$

(2)

proof of which has long been known [21]. If the first of these relations is written in partitioned form, thus:

$$[A_T^t \ A_L^t] \begin{bmatrix} C_T \\ U_L \end{bmatrix} = A_T^t C_T + A_L^t = 0,$$

(3)

it follows from Eq. (1) that

$$C_T = -B_T A_L^t.$$
This result is very useful, computationally, since it shows that each column of \( C_T \) may be calculated by taking the difference of the columns of \( B_T \) which correspond to the final and initial nodes of the corresponding link. In other words, the path-in-tree from the final node to the initial node of each link may be determined by subtracting the initial node-to-datum path from the final node-to-datum path—which is precisely what we would expect from examining the graph.

Reduced to its essentials, then, the topological characterization of a linear graph is embodied mainly in the \( A \) and \( C \) matrices and in their fundamental interrelation, \( A^T C = 0 \). The other relations shown are merely auxiliaries.

**THE ALGEBRAIC STRUCTURE ASSOCIATED WITH A LINEAR GRAPH**

The following exposition is a simplification of the abstract treatment of a network problem given by Roth [18], and is based on the algebraic topological properties of the linear graph.

The algebraic structure associated with a linear graph arises from the very simple idea of associating certain abstract mathematical quantities with the various elements of the graph. Some of these quantities may be assigned arbitrarily, while others are derived according to rules involving the matrices \( A \) and \( C \). For clarity of exposition, the only mathematical quantities to be considered explicitly will be real or complex numbers. But, later on, the very broad class of mathematical object which may enter into this algebraic structure will be identified.

We may begin by assigning an arbitrary number to each node of the graph. We designate this set of numbers by the vector \( \bar{e} \). Then, if \( \bar{e} \) is premultiplied by the augmented branch-node matrix \( \bar{A} \), this operation, or transformation, induces the assignment of a related set of numbers to the branches of the graph according to the equation

\[
e = \bar{A} \bar{e}
\]  

(5)

where the vector \( e \) now designates the set of derived branch quantities. Since each row of \( \bar{A} \) contains both \( a + 1 \) and \( a - 1 \), the derived number associated with each branch according to Eq. (5) is simply the difference between the arbitrary numbers assigned to the initial and final nodes of that branch. Specifically, for the \( i \)th branch,

\[
e_i = \sum_{j=1}^{n} \bar{a}_{ij} \bar{e}_j.
\]  

(6)

If the \( n \)-th node is allowed to serve as the datum, or reference node, it follows from Eq. (6) that

\[
e_i = \bar{a}_{in} \bar{e}_n + \sum_{j=1}^{n-1} \bar{a}_{ij} \bar{e}_j.
\]  

(7)

In terms of the elements \( a_{ij} \) of the branch-node matrix \( A \), with the column corre-
sponding to the datum node removed, Eq. (7) is equivalent to
\[ e_i = \sum_{j=1}^{n-1} a_{ij} (\bar{e}_j - \bar{e}_n). \]

It is evident from Eq. (8), therefore, that the node-to-datum differences
\((\bar{e}_j - \bar{e}_n)\) really determine the derived branch quantities \(e_i\). Accordingly, if the set of node-to-datum differences is designated by the vector \(e'\), Eq. (8) may be written in the form
\[ e = A e' \]

which is equivalent to Eq. (5).

Now if the vector \(e\) is premultiplied by the transpose of the branch-mesh matrix \(C\), this operation induces the assignment of a related set of quantities to the basic meshes of the graph. But, since \(C'tA = 0\), the mesh quantities derived in this manner are always identically zero. For it is readily seen that
\[ C'te = C'tAe' = 0 \quad e' = 0, \]

regardless of what vector \(e'\) we happen to choose.

Next, we may assign an additional arbitrary quantity to each branch, superimposing these on the derived quantities already associated with each branch by virtue of Eq. (9). We designate these arbitrarily assigned branch quantities by the vector \(E\). Premultiplication of this vector by \(C't\) then induces the assignment of nonzero quantities to the meshes, according to the relation
\[ E' = C'tE \]

where the vector \(E'\) designates the derived mesh quantities. The sum of the arbitrary and derived quantities associated with the branches may now be defined by the relation
\[ V = E + e, \]

and in view of Eqs. (10) and (11), it follows that
\[ C'tV = E'. \]

To sum up, it is possible to associate certain vectors with a linear graph in the following manner:

(a) An arbitrary vector \(e'\) may be assigned to the nondatum nodes (or rather to the node-datum pairs).

(b) A derived vector \(e = A e'\), an arbitrary vector \(E\), and their sum, \(V = E + e\), may be associated with the branches.

(c) A derived vector \(E' = C'tE\) may be associated with the meshes.

The relations between these Type 1 vectors and the topological matrices \(A\) and \(C't\) are depicted in Fig. 1, where the arrows represent the transformations defined by Eqs. (9), (10) and (11).

Now we may also associate a conjugate or dual sequence of vectors with the
elements of the graph, this time starting with the meshes and using the topological matrices in the forms $C$ and $A'$. We begin by assigning an arbitrary number to each basic mesh and we designate this set of numbers by the vector $i'$. Premultiplication of $i'$ by the branch-mesh matrix $C$ then induces the assignment of a related set of quantities to the branches according to the equation

$$i = C i'$$  \hspace{1cm} (14)

where the vector $i$ designates the set of derived branch quantities. If the vector $i$ is premultiplied by $A'$, this operation induces the assignment of a related set of quantities to the nondatum nodes of the graph. However, since $A'C = 0$, the node quantities derived in this manner are always identically zero, for

$$A'i = A'C i' = 0 \quad i' = 0,$$  \hspace{1cm} (15)

regardless of what vector $i'$ we choose.

Once again, we may assign an additional arbitrary quantity to each branch, superimposing these on the derived quantities already associated with the branches as shown by Eq. (14). We designate these arbitrarily assigned branch quantities by the vector $l$. Premultiplication of this vector by $A'$ then induces the assignment of nonzero quantities to the nondatum nodes according to the relation

$$l' = A'l.$$  \hspace{1cm} (16)

The sum of arbitrary and derived quantities associated with the branches in this sequence may now be defined as

$$J = l + i$$  \hspace{1cm} (17)

and, in view of Eqs. (15) and (16), it follows that

$$A'J = l'.$$  \hspace{1cm} (18)

To summarize once more, certain vectors may be associated with a linear graph as follows:

(a) An arbitrary vector $i'$ may be assigned to the basic meshes.

(b) A derived vector $i = C i'$, an arbitrary vector $l$, and their sum $J = l + i$, may be associated with the branches.

(c) A derived vector $l' = A'l$ may be associated with the nondatum nodes (or rather with the node-datum pairs.)

The relations between these Type 2 vectors and the topological matrices $C$ and $A'$ are shown in Fig. 2.
Equations (5) through (18) describe a rather loose algebraic structure in which the vectors in the sequence of Fig. 1 are entirely independent of those in the dual sequence of Fig. 2. Because of the looseness of this structure, a very large class of mathematical objects in addition to real and complex numbers can become part of it. Indeed, it may be shown that the elements of any additive (Abelian) group [22], [28] will satisfy Eqs. (5) through (18). However, the algebraic structure described so far does not suffice for the network problem; it requires some additional character. This is provided by introducing a correlation between the two dual sequences of vectors in Figs. 1 and 2 in the following manner:

Assume that the vector \( V \) is related to its dual vector \( J \) by the matrix transformations

\[
V = ZJ, \quad \text{or} \quad E + e = Z(I + i)
\]

(19)

and

\[
J = YV, \quad \text{or} \quad I + i = Y(E + e)
\]

(20)

where the matrices \( Z \) and \( Y \) are mutually inverse (that is \( Z = Y^{-1} \)) and have elements which are real or complex numbers. These transformations, which are depicted in Fig. 3, impose the restriction that the vectors \( e', E, i', \) and \( I \) may no longer be chosen completely independently of one another for they must now satisfy the relation

\[
E + A e' = Z(I + C i')
\]

(21)
or its equivalent

\[ I + Ci' = Y(E + Ae') \]  \hspace{1cm} (22)

which follows by combining Eqs. (9) and (14) with Eqs. (19) and (20).

In Figs. 1 and 2, the matrix multiplications involving the \( A \) and \( C \) matrices actually require only addition and subtraction of the elements of the vectors \( e' \), \( E \), \( i' \), and \( I \); this is basically why the elements of these vectors may belong to any additive (Abelian) group. But now, Eqs. (19) through (22) involve true multiplication by real or complex numbers and require, in addition, that certain associative and distributive laws be obeyed. As a consequence, the mathematical objects which may be associated with the algebraic structure of Fig. 3—which is the basic structure underlying the network problem—are the elements of any vector space [28]. This characterization is important, because vector spaces constitute a broad and significant class of mathematical object.

If Eq. (21) is premultiplied by \( C^t \) and Eq. (22) by \( A^t \), it follows that

\[ E' = C^t Z I + C^t Z C i' \]  \hspace{1cm} (23)

and

\[ I' = A^t Y E + A^t Y A e' \]  \hspace{1cm} (24)

because of Eqs. (10) and (15). Thus, if the vectors \( I \) and \( i' \) are specified, the derived vector \( E' \) is determined by Eq. (23). Similarly, if the vectors \( E \) and \( e' \) are specified, the derived vector \( I' \) is determined by Eq. (24).

The component of \( E' \) which is due to \( I \) alone may be regarded as being obtained via the composite transformation \( C^t Z \), while the component due to \( i' \) results from the transformation \( C^t Z C \). Similarly, the transformation \( A^t Y \) defines the component of \( I' \) which is due to \( E \), while \( A^t Y A \) defines that due to \( e' \).

Clearly, it is a trivial task to evaluate Eqs. (23) and (24) once the vectors \( I \), \( i' \), \( E \), and \( e' \) have been specified. But this does not correspond to what we call "the network problem." All along, we have been regarding these vectors as independent variables merely in order to simplify the description of the algebraic structure shown in Fig. 3. Actually, the network problem, in one form, treats the vectors \( E' \) and \( I' \) as independent variables and the vectors \( V \) and \( J \) as unknowns. In the other form, the vectors \( E \) and \( I \) are prescribed and the vectors \( e \) and \( i \) must be determined. These two versions of the network problem will be discussed in the following section.

THE NETWORK PROBLEM

The first version of the network problem to be described is due to Roth [18] and will be called the mathematical network problem in contrast with the more familiar electrical network problem to be discussed later. Roth's problem may be defined formally as follows:

Given: 1) a linear graph, which determines the matrices \( A \) and \( C \);

2) the transformation matrix \( Z \) and/or its inverse \( Y \); and

3) the arbitrary vectors \( E' \) and \( I' \);
find the vectors \( V \) and \( I \) such that:

1) \( V = ZJ \) and/or \( I = YV \);
2) \( A^tI = I' \); and
3) \( C^tV = E' \).

A solution to this problem may be deduced in the following way. Reverting momentarily to our previous way of interpreting Fig. 3, we note that the contribution of the vector \( i' \) to the vector \( E' \) is obtained by applying the transformation \( C^tZC \), while the contribution to \( I' \) from \( e' \) is obtained by using the transformation \( A^tYA \), as shown in Fig. 4. Now the converse is true: the contribution to \( i' \) from

\[
\begin{array}{c}
\text{Fig. 4. Roth's transformation diagram for a linear graph.}
\end{array}
\]

\( E' \) is obtained via the inverse transformation \( (C^tZC)^{-1} \), and the contribution to \( e' \) from \( I' \) is obtained via the inverse transformation \( (A^tYA)^{-1} \), also shown in Fig. 4.

Continuing in this vein, we find that the contributions to the vector \( J \) from \( E' \) and \( I' \) are given by the equation

\[
J = C(C^tZC)^{-1}E' + YA(A^tYA)^{-1}I',
\]

(25)

while the contributions to the vector \( V \) from \( I' \) and \( E' \) are given by the similar relation,

\[
V = A(A^tYA)^{-1}I' + ZC(C^tZC)^{-1}E'.
\]

(26)

Roth has shown [18] that Eqs. (25) and (26) constitute a unique solution to this problem under the (sufficient) condition that the matrix \( Z \) (or \( Y \)) is "power definite"—that is, \( Z + Z^* \) is positive definite, where \( Z^* \) is the conjugate transpose of \( Z \). He has also shown that the power definiteness of \( Z \) is sufficient to insure that the inverse transformations \( (C^tZC)^{-1} \) and \( (A^tYA)^{-1} \) exist. In a later paper [20], Roth showed that a weaker condition \( Z \) which he called "ohmicness" (see Appendix A) is sufficient for a unique solution to this problem, and is both a necessary and a sufficient condition for the existence of a solution to the more general network problem involved in Kron's method of interconnecting solutions [24], [29].

The electrical network problem, which has a slightly different flavor, may be defined in the following way:

Given: 1) a linear graph, which determines the matrices \( A \) and \( C \);

2) the transformation matrix \( Z \) and/or its inverse \( Y \); and

3) the arbitrary vectors \( E \) and \( I \);
find the vectors \( e \) and \( i \) such that:

1) \( E + e = Z (I + i) \) and/or \( I + i = Y (E + e) \);
2) \( A^t i = 0 \)
3) \( C^t e = 0 \)

In solving this problem, we need to invoke the auxiliary variables \( i' \) and \( e' \) along with the relations \( i = C i' \) and \( e = A e' \), which automatically assure that conditions 2 and 3 will be satisfied. Equations (21) and (22), which satisfy condition 1, may then be solved for the auxiliary variables \( i' \) and \( e' \). Premultiplying Eq. (21) by \( C^t \) and Eq. (22) by \( A^t \), and rearranging the results, we find that

\[
C^t (E - Z I) = C^t Z C i'
\]

and

\[
A^t (I - Y E) = A^t Y A e'.
\]

These equations may be solved for \( i' \) and \( e' \) by inverting the matrices \( C^t Z C \) and \( A^t Y A \). Hence,

\[
i' = (C^t Z C)^{-1} C^t (E - Z I)
\]

and

\[
e' = (A^t Y A)^{-1} A^t (I - Y E).
\]

Finally, Eq. (29) along with \( i = C i' \) yields

\[
i = C (C^t Z C)^{-1} C^t (E - Z I)
\]

and this, together with \( e = Z (I + i) - E \) completes the solution. In like manner, Eq. (30) along with \( e = A e' \) yields

\[
e = A (A^t Y A)^{-1} A^t (I - Y E)
\]

which, together with \( i = Y (E + e) - I \) completes the solution. Thus, there are two complementary methods (the familiar mesh and node methods) for solving this problem.

In summary, Roth's problem assumes \( E' \) and \( I' \) as given, requires \( V \) and \( J \) to be determined, and involves the inversion of both the matrices \( C^t Z C \) and \( A^t Y A \). The electrical network problem, on the other hand, assumes \( E \) and \( I \) as given, requires \( e \) and \( i \) to be determined, and may be solved by inverting either \( C^t Z C \), if \( Z \) is given, or \( A^t Y A \), if \( Y \) is given. In neither problem however, are the vectors \( e' \) and \( i' \) assigned arbitrarily as independent variables. The only purpose in making such an assignment in the discussion of the preceding section was to elucidate the algebraic properties of the network problem. Finally, as long as the \( Z \) and \( Y \) matrices consist only of real or complex numbers, the elements of the various vectors \( E, I, e, i, \) etc., involved in either version of the network problem may be elements of any vector space.

**PHYSICAL INTERPRETATIONS OF THE NETWORK PROBLEM**

Obviously, it will be possible to interpret the network problem in any given physical context only if there exists a meaningful correspondence between the
properties of the various network variables and those of the physical system in question. Interestingly enough, this correspondence does exist in a remarkably large number of instances. Indeed, the pattern of interrelations shown in Figs. 1 to 3 seems to be characteristic of much of the realm of physics. For example, the two types of network variable depicted in Figs. 1 and 2 frequently have physical counterparts in what are called across- and through-variables [15], [16].

Across-variables, which behave according to Fig. 1, characteristically sum to zero around a closed path. Through-variables, which behave according to Fig. 2, sum to zero at a point. Furthermore, these two types of variable are functionally interrelated as indicated in Fig. 3. Thus, nature provides a variety of physical situations in which the prerequisites exist for establishing network representations.

To be sure, the network description is always discrete, whereas the physical system being described by a network model may be continuous. But this only affects the quantitative accuracy of the representation and not its essential validity. Moreover, certain systems, such as electrical networks and some mechanical systems, are by nature discrete. Accordingly, they require the network mode of description.

By way of exemplifying the correspondence between the network variables and certain physical variables, we will discuss electrical, mechanical, and structural networks in the following sections.

**Electrical Networks**

The electrical interpretation of the network problem is quite straightforward and may be disposed of quickly. Briefly stated, Fig. 1 corresponds to Kirchhoff’s voltage law, while Fig. 2 corresponds to Kirchhoff’s current law. Finally, Fig. 3 introduces Ohm’s law via the transformation $Z$ (or $Y$).

In Fig. 5, the rth branch of a network is depicted in its most generalized form,

![Fig. 5. Typical branch of an electrical network.](image)

containing both a voltage source $E_r$ and a current source $I_r$, either of which may obviously be assigned any arbitrary value. From an external point of view, the voltage across and current through the branch are given as $e_r$ and $i_r$, respectively. Inside the branch, however, the actual voltage across and current through the im-
pedance element \( Z_{rr} \) (or admittance \( Y_{rr} \)) are designated by \( V_r \) and \( J_r \). With the polarity conventions shown in Fig. 5, it follows that

\[
V_r = E_r + e_r
\]

and

\[
J_r = I_r + i_r.
\]

Thus, the correspondence between these branch variables and the vectors \( E, I, e, i, V, \) and \( J \) of the previous two sections is obvious.

It should be especially noted that there are three distinct voltage variables and three distinct current variables associated with each branch of the network, and that each such variable has a unique role to play in the network problem. In particular, Ohm's law must be stated in terms of the element voltages \( V \) and element currents \( J \), since only in this way will the expression for Ohm's law remain invariant to the manner in which the branches are interconnected. More specifically, the self-impedance \( Z_{rr} \) and self-admittance \( Y_{rr} \) of the \( r \)th branch must be defined according to the following relations

\[
V_r = Z_{rr} I_r \quad \text{with} \quad J_q = 0 \quad \text{for} \quad q \neq r \quad \text{and} \quad q = 1, 2, \ldots, b
\]

\[
J_r = Y_{rr} V_r \quad \text{with} \quad V_q = 0 \quad \text{for} \quad q \neq r \quad \text{and} \quad q = 1, 2, \ldots, b
\]

Similarly, the transimpedance \( Z_{rs} \) and transadmittance \( Y_{rs} \) between the \( r \)th and \( s \)th branch must be defined by the relations

\[
V_r = Z_{rs} I_s \quad \text{with} \quad J_q = 0 \quad \text{for} \quad q \neq s \quad \text{and} \quad q = 1, 2, \ldots, b
\]

\[
J_r = Y_{rs} V_s \quad \text{with} \quad V_q = 0 \quad \text{for} \quad q \neq s \quad \text{and} \quad q = 1, 2, \ldots, b
\]

In summary, then, the element variables \( V \) and \( J \) are related by the equations

\[
V_r = \sum_{s=1}^{b} Z_{rs} I_s
\]

\[
J_r = \sum_{s=1}^{b} Y_{rs} V_s
\]

which are equivalent to Eqs. (19) and (20) and define the \( Z \) and \( Y \) matrices.

Clearly, the vector \( e' \) represents the node-to-datum voltages in terms of which the branch voltages may be computed by means of the expression \( e = A e' \). This expression is one form of Kirchhoff's voltage law. The other form of the voltage law, which corresponds to the way Kirchhoff originally expressed it, is \( C^t e = 0 \). This states that the sum of the branch voltages around each closed path is zero. Actually, these two forms of the voltage law are mathematically equivalent, since each may be shown to imply the other [17], [24].

The vector \( i' \) represents the mesh currents in terms of which the branch currents may be computed using the relation \( i = C i' \). This is one form of Kirchhoff's current law. The other form is \( A^t i = 0 \), which states that the sum of the branch currents leaving each node is zero. Again, these two forms of the current law are mathematically equivalent.
Mechanical and Structural Networks

In static mechanical or structural systems, the conjugate variables are force and displacement. Obviously, force is a through-variable, since forces, like currents, sum to zero at a point. On the other hand, displacement is an across-variable, since displacements, like voltages, sum to zero around a closed path. Finally, since force and displacement are interrelated by Hooke's law (in the ideal case), it is evident that these mechanical variables fit the pattern which is prerequisite for the existence of a network model.

Now it can be shown in detail [13] that three different force variables and three different displacement variables exist which correspond to the symbols $J$, $I$, and $i$ and $V$, $E$, and $e$ associated with each branch of the graph, as shown in Fig. 5. For example, corresponding to $I$ is an (arbitrary) applied force across each member of a structure, while corresponding to $E$ is an (arbitrary) displacement in series with each member. Corresponding to $i$ and $e$ are the force and displacement communicated to the joints to which each member is connected, while corresponding to $J$ and $V$ are the actual force and displacement experienced by the member itself.

Similarly, the symbol $i'$ in Figs. 3 and 4 corresponds to the "ring forces" and $I'$ to the applied joint forces; $e'$ corresponds to the joint displacements and $E'$ to the applied ring displacements. Finally, $Z$ corresponds to the flexibility matrix and $Y$ to the stiffness matrix for all members of the structure. Thus, Fig. 2 represents the equilibrium law in two equivalent forms, while Fig. 1 represents the compatibility law. Figure 3 depicts these two laws together with Hooke's law.

At this point, one direct benefit of the abstract treatment of the network problem shows up clearly. Even though forces and displacements are vector quantities with 2, 3 or even 6 scalar components, there is no difficulty in fitting them into the algebraic structure of the network problem, for it has already been pointed out that the quantities associated with each node, branch, or mesh of a linear graph may themselves be vectors. Thus, Eqs. (5) through (40) may be applied directly to both mechanical and structural analysis.

A precaution should be noted here, however. Since the connection of branches to a node or in a mesh implies constraints on the branch variables, it follows that all of the scalar components of every vector variable which the graph depicts as being constrained must be constrained in like manner. In other words the +1 and -1 entries in the topological $A$ and $C$ matrices must be regarded as unit matrices operating on each and every scalar component of the vector variables associated with the nodes, branches, or meshes. If the components of a particular vector variable are not all similarly constrained, then appropriate modifications must be made to the $A$ and $C$ matrices. This problem frequently arises in structural analysis [14].

One peculiar aspect of both mechanical and structural analysis is the fact that geometry as well as topology enters into the problem. For example, the numerical values of the elements of the stiffness or flexibility matrix of each structural member depend on the choice of coordinate system used to describe the member. Fortunately, this problem can be handled separately by performing appropriate geometrical transformations on the stiffness or flexibility matrices.
These transformations, in effect, convert from "local" coordinates for each member to a "global" coordinate system for the entire structure. Once this has been done, the topological analysis of the structure can be carried out without further regard to geometry. Thus, both mechanical and structural analysis can properly be regarded as within the purview of network theory.

In devising network models for dynamical systems, the traditional approach has been to make a term-by-term comparison between the differential equations describing the dynamical system and those describing a cognate electrical system. Although this practice usually works, it is by no means infallible and may even fail to lead to an analogy when one actually exists.

The inherent weakness in this approach is the fact that, in comparing the equations of performance of the dynamical systems with either the mesh or nodal equations of the electrical system, one is completely unable to recognize the differing topological character of the two types of variable involved. In other words, after the mesh or nodal equations have been compiled, no trace remains to indicate which variable sums to zero at a point and which sums to zero around a closed path. Consequently, either the mass-inductance analogy or the mass-capacitance analogy may emerge, depending on whether the mesh or nodal equations were taken as a standard of comparison.

As long as the mechanical system being modelled can be represented by a planar graph, no practical difficulty arises from using either of these two analogies. This is because of the well-known theorem that any planar graph has a dual in which the roles of the two types of network variable may be interchanged. When the graph representing the mechanical system is nonplanar, however, one of these analogies fails [2], [30].

It is not surprising to find that the mass-inductance analogy is the one which fails, since it implies that force, which sums to zero at a point, is analogous to voltage, which sums to zero around a closed path. The mass-capacitance analogy, on the other hand, always applies because it is topologically consistent in making force the analog of current and velocity the analog of voltage. For pedagogical reasons, therefore, the mass-inductance analogy, owing its existence solely to the planar graph theorem, should be discarded in favor of the mass-capacitance analogy which is fundamentally the correct one.

EXTENSION OF THE NETWORK PROBLEM

In this section we will describe a higher-dimensional homolog of the network problem associated with the familiar linear graph. In the particular case of the 3-dimensional homolog, which we will call the 3-network problem, it turns out that we must deal directly with a topological structure whose properties underlie those of the ordinary 3-dimensional space in which physical phenomena occur. Since this topological structure is invariably employed in deriving the vector calculus, it is not too surprising to discover that there exists a direct correspondence between the algebraic structure of the 3-network problem and the operational structure of the vector calculus. This correspondence will be described in the next section.
GENERALIZED NETWORKS

In discussing the 3-network problem, some new terminology will be needed. This will be introduced only as required and without detailed explanations, such as may be found in reference [23].

The generalization of the linear graph, or 1-complex, is an $n$-complex, consisting of interconnected 0-cells (points), 1-cells (line segments), 2-cells (surface elements), 3-cells (volume elements), ..., $n$-cells—all of which may be oriented, just as are the 0-cells and 1-cells of a linear graph, or 1-complex. Every $p$-cell in an $n$-complex, where $0 < p < n$, will have a set of faces which are the $(p - 1)$-cells connected (or incident) thereto. Similarly, every $p$-cell, where $0 \leq p < n$, may have (but need not necessarily have) a set of cofaces which are the $(p + 1)$-cells connected thereto.

Now the boundary of a given cell is determined by the set of all its faces while its coboundary is determined by the set of all its cofaces, with due regard being given to the orientation of the faces and cofaces relative to the cell in question. When applied to the entire $n$-complex, the boundary and coboundary operators can be expressed as matrices whose elements, like those of the $A$ and $C$ matrices for a linear graph, are $+1$, $-1$, and 0.

For example, the $A'$ matrix corresponds to, and may be regarded as a representation of, the boundary operator for all the 1-cells of a linear graph, while the $A$ matrix corresponds to the coboundary operator for all the 0-cells. Since there are no 2-cells, however, the $C$ matrix does not correspond to a boundary nor the $C'$ matrix to a coboundary operator. The explanation of this peculiarity will be given later.

An algebraic structure, similar to that depicted in Figs. 3 and 4 for the linear graph, may be associated with an $n$-complex. We may, for example, associate certain numbers with each $p$-cell of the complex, thereby defining what may be called a $p$-chain or a $p$-cochain, depending on whether it behaves according to relations like those shown in Fig. 2 or those shown in Fig. 1. When a boundary operator acts upon a $p$-chain, it produces or defines a $(p - 1)$-chain; similarly, when a coboundary operator acts upon a $p$-cochain, it defines a $(p + 1)$-cochain. In either case, the computational aspect of the transformation is quite adequately represented by a matrix equation such as $A'f = l'$ or $e = Ae'$.

In a simply-connected $n$-complex, the sequence of relations defined by the boundary operators acting on successive $n$-, $(n - 1)$-, ..., 2-, and 1-chains is called a chain sequence; the corresponding sequence of relations defined by the coboundary operators acting on successive 0-, 1-, ..., $(n - 2)$-, and $(n - 1)$-cochains is called a cochain sequence. In a multiply-connected $n$-complex, that is, one containing "holes" of various dimension, the situation is complicated somewhat by the existence of homology and cohomology groups, and the corresponding sequences are called homology and cohomology sequences [22], [23]. For example, Fig. 2 illustrates the homology sequence and Fig. 1 the cohomology sequence of a linear graph. As long as the homology and cohomology sequences are not interrelated, then the elements of each chain and cochain may belong to any additive group, as we have already pointed out. Thus, we may speak of the "group of all 2-chains" or the "group of all 0-cochains."

The transformation of, say, the 1-chains into the 0-chains, which is defined by the boundary operator, or $A'$ matrix, is called a homomorphism, or many-to-one correspondence. Similarly, the coboundary operator, or $A$ matrix, which trans-
forms 0-cochains into 1-cochains defines another homomorphism. On the other hand, the transformation \( Z \) (or \( Y \)) between the 1-chains and 1-cochains, in Figs. 3 and 4, is called an *isomorphism* since it defines a reversible 1-to-1 correspondence.

Now any \( p \)-chain whose boundary is zero is called a \( p \)-cycle, and if it also happens to be the boundary of a \((p + 1)\)-chain, it is called a *bounding* \( p \)-cycle. Not every \( p \)-cycle is a bounding \( p \)-cycle, however, particularly in a multiply-connected \( n \)-complex. For example, in a linear graph, every \( i' \) vector corresponds to a 1-cycle. But since there are no 2-cells, and hence no 2-chains, there can be no bounding 1-cycles.

A special name is given, therefore, to the group which contains all non-bounding \( p \)-cycles, namely, the \( p \)th homology group. Accordingly, in Fig. 2, the symbol \( i' \) represents the first homology group, and the matrix \( C \) corresponds to the "natural" homomorphism (or transformation) of this group into the group of 1-chains [18]. Similarly, any \( p \)-cochain whose coboundary is zero is called a \( p \)-cocycle, and if it is also a coboundary of a \((p - 1)\)-cochain, it is called a *cobounding* \( p \)-cocycle. The group which contains all non-cobounding \( p \)-cocycles is called the \( p \)th cohomology group. In Fig. 1, the symbol \( E' \) corresponds to the first cohomology group and the matrix \( C' \) corresponds to the natural homomorphism of the group of 1-cochains into this group [18].

When the homology and cohomology sequences are interrelated by an isomorphism, as shown in Figs. 3 and 4, then the *components* of each chain and cochain, or each homology and cohomology group become restricted to vector spaces [28], as we have stated before. Indeed, the chains, cochains, etc., are themselves elements of vector spaces, and so we may speak of the "space of all 1-chains" or the "space of all 0-cochains," etc.

If the \( n \)-complex is simply-connected, the homology and cohomology groups do not enter the picture, since all cycles are bounding cycles and all cocycles are cobounding cocycles. Thus, the only homomorphisms that need to be considered in the chain sequence are those defined by the boundary operators, and in the cochain sequence, those defined by the coboundary operators. This is the only case we will consider in any detail.

One final word about the boundary and coboundary operators: It is an easily proved theorem that "the boundary of the boundary is zero" and also that "the coboundary of the coboundary is zero." This basic theorem is true for essentially the same reason that \( A^t C = 0 \), and it holds for all adjacent dimensionalities of an \( n \)-complex [22], [23].

We shall now consider the algebraic topological structure of a simply-connected 3-complex (or 3-network), with its chain and cochain sequences interrelated by two isomorphisms, as shown in Fig. 6. We let \( Q_0 \), \( Q_1 \), \( Q_2 \) and \( Q_3 \) represent the 0-, 1-, 2- and 3-chains, with \( C_{01}, C_{12} \) and \( C_{23} \) being the connection matrices, or boundary operators between the chains of adjoining dimension. Similarly, we let \( Q^0, Q^1, Q^2, \) and \( Q^3 \) represent the 0-, 1-, 2- and 3-cochains, with \( C^{10}, C^{21}, \) and \( C^{32} \) being the coboundary operators. As in the case of the linear graph, the coboundary operators may be represented by matrices which are the transposes of the corresponding matrices representing the boundary operators.

Accordingly, we have

\[
C^{10} = C_{01}^t, \quad C^{21} = C_{12}^t, \quad \text{and} \quad C^{32} = C_{23}^t, \quad (41)
\]
so that the basic theorem regarding the boundary and coboundary operators for a 3-complex may be written as follows:

\[ C_{01} C_{12} = 0 \quad \text{and/or} \quad C_{12} C_{01}^t = 0 \]  

(42)

and

\[ C_{12} C_{23} = 0 \quad \text{and/or} \quad C_{23} C_{12}^t = 0. \]  

(43)

If we assume that there is an isomorphism \( S_1 \) between the space of 1-chains \( Q_1 \) and the space of 1-cochains \( Q^1 \), with \( T_{1} \) being the inverse of \( S_1 \), then the induced isomorphism between the space of 0-cochains \( Q^0 \) and the space of 0-chains \( Q_0 \) is \( C_{01} T_1 C_{10}^t \), as shown in Fig. 6. Using Eq. (41), we see that this is identical to the congruence transformation \( C_{01} T_1 C_{01}^t \) shown in Fig. 7. In like manner, if we assume that \( S_2 \) is an isomorphism between the space of 2-chains \( Q_2 \) and the space of 2-cochains \( Q^2 \), with \( T_2 \) its inverse, then the induced isomorphism between \( Q_3 \) and \( Q^3 \) is \( C_{32} S_2 C_{23} \), or \( C_{23} S_2 C_{23} \). In Appendix A, it is shown by the application of Roth's theorem [18], [19] that if \( S_1 \) and \( S_2 \) are "ohmic," both of the inverse transformations \( (C_{01} T_1 C_{01}^t)^{-1} \) and \( (C_{23} S_2 C_{23})^{-1} \) exist.

Now every nonzero 1-chain in \( Q_1 \) (or 2-chain in \( Q_2 \)) whose boundary is zero belongs to the subspace of all 1-cycles (or 2-cycles) designated by \( M_1 \) (or by \( M_2 \)).
in Fig. 7. All 1-chains (or 2-chains) which are not 1-cycles (or 2-cycles) are designated by \( P_1 \) (or by \( P_2 \)) and occupy the entire space \( Q_1 \) (or \( Q_2 \)).

The subspace \( M_1 \) is the image of all of \( Q_2 \) under the transformation \( C_{12} \); but only that part of \( P_2 \) which is exterior to \( M_2 \) makes any contribution to \( M_1 \) under this transformation, since \( M_2 \) is mapped into zero. \( M_1 \), in turn, is mapped into zero by \( C_{01} \)—since the "boundary of the boundary is zero"—and is said to be the kernel of \( C_{01} \). Similarly, \( M_2 \) is the kernel of \( C_{12} \) and the image of \( P_3 \) under \( C_{23} \). \( M_0 \) is the image of \( Q_1 \) under \( C_{01} \).

Like designations are used in Fig. 7 for the subspaces \( M^1 \) and \( M^2 \) of 1- and 2-cocycles and for 1- and 2-cochains \( P^1 \) and \( P^2 \) which are not cocyclic. \( M^1 \) and \( M^2 \) are the kernels of \( C^1_{12} \) and \( C^1_{23} \), respectively, and the images of \( P^0 \) under \( C^0_{01} \) and of \( Q^1 \) under \( C^1_{12} \), respectively. \( M^3 \) is the image of \( Q^2 \) under \( C^1_{23} \).

The dimensions of these various spaces and subspaces are readily computed topological invariants determined by the ranks \( p_1 \), \( p_2 \) and \( p_3 \) of the connection matrices \( C_{01} \), \( C_{12} \), and \( C_{23} \). In order to compute \( p_1 \), \( p_2 \) and \( p_3 \), the connectivity numbers [21] (or Betti numbers) \( R_0 \), \( R_1 \), \( R_2 \) and \( R_3 \) are required. \( R_0 \) is the number of subcomplexes, equal to \( k \), the number of subgraphs or separate parts in our earlier discussion of linear graphs. \( R_p - 1 \) is the dimension of the \( p \)th homology group and is equal to the maximal number of linearly independent nonbounding \( p \)-cycles. In a linear graph, for example, \( R_1 - 1 = m \), the number of basic meshes.

If we let \( n_0 \), \( n_1 \), \( n_2 \) and \( n_3 \) be the numbers of 0-, 1-, 2-, and 3-cells in a 3-complex, then it may be shown [21] in general that

\[
\begin{align*}
    p_1 &= n_0 - R_0, \\
    p_2 &= n_1 - p_1 - (R_1 - 1), \\
    p_3 &= n_2 - p_2 - (R_2 - 1) = n_3 - (R_3 - 1).
\end{align*}
\]

For a simply-connected 3-complex, these relations reduce to

\[
\begin{align*}
    p_1 &= n_0 - 1, \\
    p_2 &= n_1 - p_1, \\
    p_3 &= n_2 - p_2 = n_3.
\end{align*}
\]

The connection matrix \( C_{01} \), which is identical with the \( A^t \) matrix for a linear graph, has only \( p_1 = n_0 - 1 \) rows, the row corresponding to the datum node having been deleted for the same reason that it was in \( A^t \). Thus, \( C_{01} \) is a \( p_1 \times n_1 \) matrix and its rank is \( p_1 \). The matrices \( C_{12} \) and \( C_{23} \) are of dimension \( n_1 \times n_2 \) and \( n_2 \times n_3 \), with ranks \( p_2 \) and \( p_3 \), respectively, given by Eqs. (48) and (49).

Since the datum node is in effect disregarded, all 0-chains and 0-cochains will contain only \( p_1 = n_0 - 1 \) components; hence the spaces \( Q_0 \) and \( Q^0 \) of Fig. 6 (relabelled \( M_0 \) and \( P^0 \) in Fig. 7) are of dimension \( p_1 \). However, the 1-, 2-, and 3-chains and cochains will contain \( n_1 \), \( n_2 \), and \( n_3 \) components, so that the dimensions of \( Q_j \) and \( Q^j \) are \( n_j \), \( j = 1, 2, 3 \).

Suppose that a particular 2-chain, belonging to the space \( Q_2 \) and having \( n_2 \) components, happens to be a 2-cycle. It will therefore be confined to the subspace of all 2-cycles, namely \( M_2 \). Now this subspace is characterized by the relation \( C_{12} M_2 = 0 \) which represents a total of \( p_2 \) constraints on the \( n_2 \) components of every vector in \( M_2 \). Since these constraints are linear dependencies, it fol-
allows that the maximum number of linearly independent components of a vector in $M_2$ must be $n_2 - p_2 = p_3$. Hence, the dimension of $M_2$ is $p_3$. By a similar argument, the dimensions of all the other subspaces may be determined, as summarized in Table I.

Table I—Subspace Dimensions for 3-Complex

<table>
<thead>
<tr>
<th>Subspace</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_0, P^0, M^1$</td>
<td>$p_1 = n_0 - 1$</td>
</tr>
<tr>
<td>$M_1, M^2$</td>
<td>$p_2 = n_1 - p_1$</td>
</tr>
<tr>
<td>$M_2, P_3, M^3$</td>
<td>$p_3 = n_2 - p_2 = n_3$</td>
</tr>
</tbody>
</table>

Corresponding to the two Kirchhoff laws for a linear graph, each law being expressed in two equivalent forms, are four such laws for a 3-complex:

\[ C_{01} M_1 = 0 \quad \text{and/or} \quad M_1 = C_{12} P_2, \]  
(50a, 50b)

\[ C_{12} M_2 = 0 \quad \text{and/or} \quad M_2 = C_{23} P_3, \]  
(51a, 51b)

\[ C'_{12} M^1 = 0 \quad \text{and/or} \quad M^1 = C'_{01} P^0, \]  
(52a, 52b)

\[ C'_{23} M^2 = 0 \quad \text{and/or} \quad M^2 = C'_{12} P^1. \]  
(53a, 53b)

Similarly, there are two pairs of expressions corresponding to Ohm’s law:

\[ M_1 + P_1 = T_1 (M^1 + P^1), \]  
(54a)

and/or

\[ M^1 + P^1 = S_1 (M_1 + P_1); \]  
(54b)

also

\[ M_2 + P_2 = T_2 (M^2 + P^2), \]  
(55a)

and/or

\[ M^2 + P^2 = S_2 (M_2 + P_2). \]  
(55b)

The 3-Network Problem

The author originally expected to find two versions of the 3-network problems analogous to Roth’s problem and the electrical network problem for a linear graph. However, every attempt to set up and solve a Roth-type problem, with $M_0$ and $M^3$ specified, has failed—apparently because such a problem is underdetermined.

The only 3-network problem which the author has been able to define and solve corresponds, roughly, to the electrical network problem—but with notable differences. Specifically, one cannot assume arbitrary vectors $P^1, P_1, P^2$, and $P_2$ since $P^1$ and $P_2$ are determined by $P_1$ and $P^2$. The 3-network problem, then, may be stated formally as follows:

Given: 1) a simply-connected 3-complex and its connection matrices $C_{01}, C_{12}$ and $C_{23};$

2) the isomorphisms $S_1$ and $S_2$ and/or their inverses $T_1$ and $T_2;$ and

3) the arbitrary vectors $P_1$ and $P^2;$

find the vectors $M_1, M_2, P_2, P_3$ and $P^0, P^1, M^1, M^2$ such that Equations 50 through 55 hold true.
To solve this problem, we proceed as in the derivation of Eqs. (27) and (28). First, we premultiply Eq. (54a) by $C_{01}$ and use Eqs. (50a) and (52b) to obtain the result

$$ C_{01}(P_1 - T_1 P^1) = C_{01} T_1 C_{01} P^0. \quad (56) $$

Next, we premultiply Eqs. (54b), (55a), and (55b) by $C_{12}'$, $C_{12}$, and $C_{23}'$, respectively, and use the appropriate relations from Eqs. (50) through (53) to derive the following expressions:

$$ C_{12}(P_1 - S_1 P_1) = C_{12}' S_1 C_{12} P_2, \quad (57) $$
$$ C_{12}(P_2 - T_2 P^2) = C_{12} T_2 C_{12}' P^1, \quad (58) $$
$$ C_{23}'(P^2 - S_2 P_2) = C_{23}' S_2 C_{23} P_3. \quad (59) $$

Now Eqs. (57) and (58) can be solved simultaneously for $P^1$ and $P_2$. Assuming that $S_1$ and $S_2$ are ohmic, we may show that the inverse transformations required, namely $(C_{12}' S_1 C_{12})^{-1}$ and $(C_{12} T_2 C_{12}')^{-1}$, exist. (See Appendix A.) Hence, we may write

$$ P_2 = (C_{12}' S_1 C_{12})^{-1} C_{12}'(P_1 - S_1 P_1) \quad (60) $$

and

$$ P^1 = (C_{12} T_2 C_{12}')^{-1} C_{12}(P_2 - T_2 P^2). \quad (61) $$

By substituting Eq. (60) into (58), we get

$$ C_{12}(C_{12}' S_1 C_{12})^{-1} - T_2 C_{12}' P^1 - C_{12}' (C_{12}' S_1 C_{12})^{-1} C_{12}' S_1 P_1 + C_{12} T_2 P^2. \quad (62) $$

Similarly, by substituting Eq. (61) into (57), we get

$$ C_{12}'(C_{12} T_2 C_{12}')^{-1} - S_1 C_{12} T_2 P^2 - C_{12}' (C_{12} T_2 C_{12}')^{-1} C_{12} T_2 P^2 + C_{12}' S_1 P_1. \quad (63) $$

Thus $P^1$ and $P_2$ are functions of $P_1$ and $P^2$. The converse is not true, however, since Eq. (57) cannot be solved for $P_1$, nor can Eq. (58) be solved for $P^2$. To solve for $P^1$ and $P_2$, then, the appropriate coefficient matrices must be non-singular. This imposes an additional condition on $S_1$ and $S_2$, namely that both of the transformations

$$ [(C_{12}' S_1 C_{12})^{-1} - T_2] \quad \text{and} \quad [(C_{12} T_2 C_{12}')^{-1} - S_1] $$

must now be ohmic. If this condition is met, the needed inverses exist, and we may write the solutions for Eqs. (62) and (63) as follows:

$$ P^1 = C_{12}'(C_{12}' S_1 C_{12})^{-1} - T_2 C_{12}'^{-1} C_{12}[T_2 P^2 + (C_{12}' S_1 C_{12})^{-1} C_{12}' S_1 P_1] \quad (64) $$

and

$$ P_2 = C_{12}'[(C_{12} T_2 C_{12}')^{-1} - S_1] C_{12}^{-1} C_{12}'[S_1 P_1 + (C_{12} T_2 C_{12}')^{-1} C_{12} T_2 P^2]. \quad (65) $$

With $P^1$ and $P_2$ thus determined, $M^2$ is easily calculated from Eq. (53b) and $M_1$ from Eq. (50b). To solve for $M^1$, we need $P^0$, and to solve for $M_2$, we need to obtain $P_3$. The values of these two auxiliary variables can be gotten from Eqs. (56) and (59). Assuming that $S_1$ and $S_2$ are ohmic, the inverse matrices $(C_{01} T_1 C_{01})^{-1}$ and $(C_{23}' S_2 C_{23})^{-1}$ may be shown to exist (see Appendix A), so that we may write

$$ P^0 = (C_{01} T_1 C_{01})^{-1} C_{01}(P_1 - T_1 P^1). \quad (66) $$
and

\[ P_3 = (C_{23}^t S_2 C_{23})^{-1} C_{23}^t (P^2 - S_2 P_2). \]  \hspace{1cm} (67)

Equations (52b) and (51b) then yield \( M^1 \) and \( M_2 \) directly. Finally, \( M_0 \) and \( M^3 \) can be evaluated from the relations \( M_0 = C_{01} P_1 \) and \( M^3 = C_{23}^t P^2 \).

Specifying \( M_0 \) and \( M^1 \) alone as independent variables is insufficient to determine a bona fide 3-network problem. This may be shown as follows: using Eqs. (52b), (64), and (66), we may write

\[ Q^1 M^1 \cdot P^1 C_{01}^t (C_{01} T_1 C_{01})^{-1} M_0 + FS_1 P_1 + GT_2 P^2 \]  \hspace{1cm} (68)

where \( F \) and \( G \) represent very complicated expressions, the precise form of which is not germane to our argument. Similarly, from Eqs. (51b), (65), and (67), we obtain the relation

\[ Q_2 M_2 \cdot P_2 C_{23}^t (C_{23}^t S_2 C_{23})^{-1} M^3 + HS_1 P_1 + KT_2 P^2 \]  \hspace{1cm} (69)

where \( H \) and \( K \) are again complicated expressions. Thus, \( Q^1 \) is a function not only of \( M^3 \) but also of \( P_1 \) and \( P^2 \), while \( Q_2 \) is a function of \( M^3 \) and of \( P_1 \) and \( P^2 \) as well.

Since the transformations \( M_0 = C_{01} P_1 \) and \( M^3 = C_{23}^t P^2 \) are many-to-one, there are an infinity of vectors \( P_1 \) and \( P^2 \) which correspond to any given vectors \( M_0 \) and \( M^3 \). To each such choice of \( P_1 \) and \( P^2 \), therefore, there will correspond a different set of vectors \( Q^1 \) and \( Q_2 \) according to Eqs. (68) and (69). In other words, the 3-network problem is underdetermined when only \( M_0 \) and \( M^3 \) are specified.

THE 3-NETWORK PROBLEM AND THE VECTOR CALCULUS

In this section, we will show how the operational structure of the vector calculus corresponds to the algebraic structure of the 3-network problem. Basically, the topological properties of a 3-complex are (non-metric) properties of the Euclidean 3-space which the vector calculus seeks to describe. Therefore, it is not too surprising to find that the vector calculus is what emerges from the algebraic description of the 3-network problem when the underlying topological structure passes to the limit of zero mesh size.

Recognizing this intimate relationship between topology and the vector calculus enables us to understand more clearly why network models exist for several important classes of partial differential equation that devolve from the vector calculus. Moreover, it sheds light on the interrelations between the gradient, curl, and divergence operators by organizing them into a meaningful structural pattern.

The author has found this pattern to be especially helpful in clarifying the interplay amongst the variables in Maxwell's equations for the electromagnetic field. Moreover, the precise manner in which the electrodynamic and magneto-dynamic fields are coupled together becomes easy to identify and understand in terms of the algebra of the 3-network problem.

After explaining the relationship between the 3-network problem and the vector calculus, we will discuss the electric and magnetic field equations, for both the static and dynamic cases, in terms of this relationship. First, however, it is
necessary to introduce the concept of a dual 3-complex in order to prepare the way for a necessary modification of the algebraic structure of Fig. 7.

The Duality Concept and Its Consequences

It is well known that the dual of any planar graph exists [26], [27] in which each mesh of the primal graph corresponds with a node of the dual, and vice versa. In order to establish this correspondence, it is required to place an extra node of the dual graph exterior to the primal graph; this node may be regarded as the datum. There will also be a node of the primal graph exterior to the dual graph, and this node may serve as the primal datum.

The concept of duality may be extended to higher-dimensional complexes (see reference [21], p. 92) whenever it is possible to set up a 1-to-1 correspondence between the $k$-cells of the primal $n$-complex and the $(n-k)$-cells of the dual $n$-complex. The extension to dual 2-complexes, for example, is immediate on the basis of the planar graph duality property, since each mesh of a planar graph may be identified with a 2-cell of a 2-complex having the same 1-skeleton, or linear graph configuration.

For our present purpose, we will consider the dual of a simply-connected cubical 3-complex, as depicted in Fig. 8 where the primal complex $K$ is drawn in

![Diagram of primal and dual cubical 3-complexes]

Fig. 8. Primal and dual cubical 3-networks.
light lines while the dual complex $K$ is drawn in heavy lines. Clearly, these two complexes are so disposed that the 0-cells of $K$ correspond to 3-cells of $K$, 1-cells of $K$ to 2-cells of $K$, etc., at least to some extent, in the drawing.

These complexes may, of course, be extended ad infinitum by adding more cells. However, to establish a complete correspondence between all the $k$-cells of $K$ and the $(3 - k)$-cells of $K$ in a finite complex, we may resort to the following device. First, we surround the entire configuration of Fig. 8 with a spherical surface $S$ (not shown) from which a single point has been removed at, say, its south pole. We then connect the ends of the outermost 1-cells of $K$ to this surface and “fill in” the additional 2-cells and 3-cells of $K$ required to complete the correspondence with all the outermost 1-cells and 0-cells of $K$, except for the particular 0-cell of $K$ which is in direct apposition to the deleted point at the south pole of $S$. This 0-cell will be taken as the datum of $K$.

Since $S$ is homeomorphic to a point, we may continuously deform it, along with all the 1-, 2-, and 3-cells of $K$ connected thereto, until $S$ merges into a single 0-cell at, say, its north pole. This 0-cell, which is exterior to $K$ and to which all the outermost 1-, 2-, and 3-cells of $K$ are now connected, is taken as the datum of $K$. Thus, the dual correspondence is complete with all the 3-cells of $K$ corresponding to nondatum 0-cells of $K$, and conversely, and with the datum of $K$ exterior to $K$, and conversely.

By virtue of the 1-to-1 correspondence between the $k$-cells of $K$ and the $(3 - k)$-cells of $K$, two important relationships emerge: one associated with the topological properties of these dual complexes, and the other associated with their algebraic structures. First, a duality exists between the boundary operators of $K$ and the coboundary operators of $K$. In terms of the connection matrices, it is clear that $C_{01}$, which depicts the connection between the (nondatum) 0-cells and the 1-cells of $K$, must be identical with the matrix $C^{32}$ which describes the connections between the 3-cells and 2-cells of $K$. Thus, in view of Eq. (41),

$$C_{01} = C^{32} = C'^{t}_{23}.$$  \hspace{1cm} (70)

Similarly, it follows that

$$C_{12} = C^{21} = C^{t}_{12}$$  \hspace{1cm} (71)

and

$$C_{23} = C^{10} = C^{t}_{01}.$$  \hspace{1cm} (72)

In reference [21], incidentally, the definition of duality is based on the identity of the primal and dual connection matrices in the manner shown by Eqs. (70), (71), and (72).

The second consequence of this duality is the fact that the $k$-chains of $K$ are isomorphic with the $(3 - k)$-cochains of $K$ and conversely. Accordingly, we may substitute the cochain sequence of $K$ in place of the chain sequence of $K$. Moreover, in view of Eqs. (70)-(72), we may do so without even replacing the connection matrices $C_{01}$, $C_{12}$, and $C_{23}$. The result of this substitution is shown in Fig. 9. Obviously, we can apply Eqs. (50) through (69) directly to this dual 3-network problem simply by substituting the symbols $P^0$, $M^1$, $P^1$, $M^2$, $P^2$, and $M^3$, in place of the symbols $P_3$, $M_2$, $P_2$, $M_1$, $P_1$, and $M_0$, respectively, wherever they appear.
It is this algebraic structure of the dual 3-network problem that is needed for the following discussion of the vector calculus. A similar structure consisting of two chain sequences could also be established, but this has no bearing on what follows.

**Operational Structure Underlying the Vector Calculus**

Interestingly enough, the vector calculus follows the pattern of a cochain sequence rather than that of a chain sequence. To show this, we assume that an arbitrary scalar field \( \phi(x, y, z) \) is defined throughout a simply-connected region of 3-space which is subdivided by a finite cubical 3-complex \( K \). Let \( \phi(x, y, z) \) be evaluated at each 0-cell of \( K \), thereby defining a 0-cochain belonging to \( P^0 \). Then the line integral of \( \text{grad} \phi \) between two adjacent 0-cells, \( a \) and \( b \), is

\[
\int_a^b \text{grad} \phi \cdot d\Gamma = \phi(b) - \phi(a). \tag{73}
\]

This defines a number assignable to the 1-cell between \( a \) and \( b \) in precisely the same way as that implied by the relation \( e = \overline{\overline{F}} \) in Eq. (5) of our earlier discussion of the linear graph problem. If we subtract from \( \phi(x, y, z) \) its value at the datum node of \( K \), the resulting scalar field will correspond to node-to-datum differences, and Eq. (73) will then be analogous to \( e = A \overline{\overline{F}} \) or to Eq. (52b). (For the sake of simplicity, we will always assume that this has been done—or that the value of \( \phi \) at the datum is zero.)

If we apply the line integral of Eq. (73) to each 1-cell of \( K \), we define a 1-cochain belonging to \( M^1 \). For infinitesimal 1-cells, the elements of this 1-cochain will be \( -\text{grad} \phi \cdot d\Gamma \), where \( d\Gamma \) is the vector representation of the 1-cell. Thus, the line integral of \( \text{grad} \phi \) acts like the coboundary operator \( C^1_0 \) for 1-cells of finite size, whereas the (negative) gradient operator itself acts like \( C^0_0 \) in the limiting case of vanishingly small 1-cells.

We next assume the existence of an arbitrary vector field \( \overline{\overline{A}}(x, y, z) \) throughout the region subdivided by \( K \). By means of the line integral of \( \overline{\overline{A}} \) along each 1-cell of \( K \), we define a 1-cochain which belongs to the entire space \( Q^1 \), and which we
designate by the symbol $P^1$. For infinitesimal 1-cells, the elements of $P^1$ will be $\overrightarrow{\mathbf{A}} \cdot d\overrightarrow{1}$. Using Stoke's theorem

$$\oint \overrightarrow{\mathbf{A}} \cdot d\overrightarrow{1} = \int_S \text{curl} \overrightarrow{\mathbf{A}} \cdot d\overrightarrow{s}, \quad (74)$$

and integrating $\overrightarrow{\mathbf{A}} \cdot d\overrightarrow{1}$ around the boundary of each 2-cell of $K$, we define a 2-cochain belonging to the subspace $M^2$. For infinitesimal 2-cells, the elements of this 2-cochain will be $\text{curl} \overrightarrow{\mathbf{A}} \cdot d\overrightarrow{s}$, where $d\overrightarrow{s}$ is the vector representation of the 2-cell. Thus, the line integral of $\overrightarrow{\mathbf{A}}$ around the boundary of each finite 2-cell acts like the coboundary operator $C^t_{12}$, whereas the curl operator itself acts like $C^t_{12}$ in the limiting case of vanishingly small 2-cells.

Finally, we assume that another arbitrary vector field $\overrightarrow{\mathbf{B}}(x, y, z)$ exists in the region of $K$ and, by means of the surface integral of $\overrightarrow{\mathbf{B}}$ over each 2-cell of $K$, we define a 2-cochain. This 2-cochain belongs to the entire space $Q^2$ and is designated by the symbol $P^2$. For infinitesimal 2-cells, the elements of $P^2$ will be $\overrightarrow{\mathbf{B}} \cdot d\overrightarrow{s}$.

Using Gauss' theorem,

$$\int_S \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{s} = \int_V \text{div} \overrightarrow{\mathbf{B}} \ dv, \quad (75)$$

and integrating $\overrightarrow{\mathbf{B}} \cdot d\overrightarrow{s}$ over the boundary of each 3-cell of $K$, we define a 3-cochain belonging to $M^3$. For infinitesimal 3-cells, the elements of this 3-cochain will be $\text{div} \overrightarrow{\mathbf{B}} \ dv$, where $dv$ is the volume of the 3-cell. Thus, the surface integral of $\overrightarrow{\mathbf{B}}$ over the boundary of each finite 3-cell acts like the coboundary operator $C^t_{23}$, whereas the divergence operator itself acts like $C^t_{23}$ in the limiting case of vanishingly small 3-cells.

Now the cochains $P^0$, $P^1$, and $P^2$, being defined in terms of the arbitrary functions $\phi$, $\overrightarrow{\mathbf{A}}$, and $\overrightarrow{\mathbf{B}}$, are themselves arbitrary. On the other hand, the cochains $M^1$, $M^2$, and $M^3$, being defined in terms of the derived functions $\text{grad} \phi$, $\text{curl} \overrightarrow{\mathbf{A}}$, and $\text{div} \overrightarrow{\mathbf{B}}$, are not arbitrary. Moreover, the cochains $M^1$ and $M^2$ are images under the coboundary operators $C^t_{01}$ (or $\text{grad}$) and $C^t_{12}$ (or $\text{curl}$), and so are kernels of $C^t_{12}$ (or $\text{curl}$) and $C^t_{23}$ (or $\text{div}$), respectively.

In other words, if we take the line integral of $\text{grad} \phi$ around the boundary of each 2-cell of $K$, the 2-cochain so defined is zero. But this operation is tantamount to the operation $C^t_{12}M^1$ whose result vanishes because of the topological principle that the coboundary of the coboundary is zero. Even for 2-cells where the path of integration is vanishingly small, the topological nature of this operation remains unchanged and is inherent in the point relation

$$\text{curl} \overrightarrow{\text{grad}} \phi = 0. \quad (76)$$

Thus, the fundamental operator relation $\text{curl} \overrightarrow{\text{grad}} = 0$ is simply the vector calculus counterpart of the topological expression $C^t_{12}C^t_{01} = 0$.

Similarly, if we take the surface integral of $\text{curl} \overrightarrow{\mathbf{A}}$ over the boundary of each 3-cell of $K$, the 3-cochain so defined is zero—again because the coboundary of the coboundary is zero. This fact is reflected in the point relation

$$\text{div} \text{curl} \overrightarrow{\mathbf{A}} = 0 \quad (77)$$

and in the operator equation $\text{div} \text{curl} = 0$, which is the counterpart of $C^t_{23}C^t_{12} = 0$. 
These relationships are summarized in Fig. 10 wherein

1) $\phi$ is an arbitrary scalar potential function;
2) $\vec{V} = \vec{A} - \text{grad} \phi$, with $\vec{A}$ an arbitrary vector potential function;
3) $\vec{W} = \vec{B} + \text{curl} \vec{A}$, with $\vec{B}$ another arbitrary vector potential function;
4) $\omega = \text{div} \vec{B}$, a derived scalar function.

It is interesting to observe that there are two distinct types of vector depicted in this diagram: namely, a vector such as $\vec{V}$ which is related to the 1-cells and 1-cochains of $K$, and a vector such as $\vec{W}$ which is related to the 2-cells and 2-cochains of $K$. These vectors are designated as line density and surface density variables, and they appear to correspond to polar and axial vectors.

As Fig. 10 indicates, the cochain sequence that obtains for a 3-complex of finite cells goes over, in the limit of infinitesimal cell size, to a virtual cochain sequence appropriate to the vector calculus. If we assume the existence of a dual 3-complex $\overline{K}$ having a conjugate set of scalar and vector variables $\varphi, \vec{U}, \vec{J}$, and $\rho$, and then let the 1-, 2-, 3-cells of $K$ pass to the limit of zero size, we obtain the dual cochain sequence shown in the upper part of Fig. 11. Finally, we can interrelate the primal and dual cochain sequences of Fig. 11 by means of constitutive parameters, such as $z, y, s, \text{ and } t$, defined by the relations

$$\vec{V} = z \vec{J} \quad \text{and/or} \quad \vec{J} = y \vec{V} \quad (78)$$

and

$$\vec{W} = s \vec{U} \quad \text{and/or} \quad \vec{U} = t \vec{W}, \quad (79)$$

where $y = z^{-1}$ and $t = s^{-1}$.

Figure 11 illustrates the operational structure that underlies the vector calculus when the latter is used to characterize some physical phenomenon, e.g., electromagnetism. The constitutive parameters $s, t, x, \text{ and } y$ are used to describe
certain properties of the medium in which the phenomenon under consideration occurs. Broadly speaking, these parameters may be scalar constants (or functions of position) for isotropic media or tensor quantities for anisotropic media.

Since Fig. 11 is the direct counterpart of Fig. 9, which depicts the algebraic structure of the dual 3-network problem, we will interpret Fig. 11 in terms of Eqs. (50) through (69). We will transcribe these equations by substituting \(-\nabla f\) for \(C_{01}\) or \(C_{23}\), curl for \(C_{12}\) or \(C_{12}\), div for \(C_{23}\) or \(C_{01}\), etc., and inserting the appropriate vector or scalar variables in place of \(M_0, P^1\), etc. The nature of these substitutions should be clear from the similarity of the transformation diagrams themselves.

The electromagnetic field has been chosen as a vehicle for interpreting Fig. 11 because it has certain inherent features which are particularly interesting from a topological point of view. Moreover, it blends nicely into the subsequent discussion of network models for partial differential equations of a general type.

**THE ELECTROMAGNETIC FIELD EQUATIONS**

In treating the electromagnetic field equations on the basis of Fig. 11 and the dual 3-network problem, we will deal with the electrostatic and magnetostatic fields first. Once these are understood from a topological point of view, it is a simple matter to extend the treatment to the case of electrodynamic and magnetodynamic fields, and then to show how the corresponding operational structures merge to form that of the electromagnetic field. This approach will clarify the topological character of the displacement current, which is the keystone in the development of Maxwell's equations.

**The Electrostatic Field Problem**

Only a subset of the operational structure of Fig. 11 is required to describe the electrostatic field problem, as shown in Fig. 12. It is interesting to note that the physical dimensions of the variables involved are consistent with their assignment as 0-, 1-, 2-, or 3-cochains. Indeed, dimensional consistency requires that the concept of a dual 3-complex be invoked.
The electrostatic variables and their dimensions are:

φ, the electric potential in volts,

\( \mathbf{E} \), the electric field strength in volts/meter,

\( \mathbf{D} + \mathbf{d} \), the displacement in coulombs/(meter)^2,

\( \rho \), the charge density in coulombs/(meter)^3.

To be precise, we have specified the displacement as the sum of two components, \( \mathbf{D} \) and \( \mathbf{d} \). The component \( \mathbf{D} \) corresponds to the symbol \( P^2 \) in Fig. 9, which represents any dual 2-cochain occupying the entire space of 2-cochains. The component \( \mathbf{d} \), on the other hand, corresponds to the symbol \( M^2 \), which represents the subspace of 2-cocycles. In more familiar terms, \( \mathbf{d} \) signifies the solenoidal component of displacement, having zero divergence.

The basic relations analogous to Eqs. (50) to (55) are:

\[
\text{div } \mathbf{d} = 0 \tag{80}
\]

\[
\rho = \text{div } \mathbf{D} \tag{81}
\]

\[
\mathbf{E} = -\text{grad } \phi \tag{82}
\]

and

\[
\mathbf{D} + \mathbf{d} = \varepsilon \mathbf{E} \tag{83}
\]

where \( \varepsilon \) is the dielectric constant. In view of Eq. (80), it is clear that there exists a vector potential \( \mathbf{h} \), having the dimensions of coulombs/meter, such that

\[
\mathbf{d} = \text{curl } \mathbf{h} \tag{84}
\]
as shown in Fig. 12. This vector potential is not needed for solving electrostatic field problems, but recognizing its existence is important from a topological standpoint.

We may obtain an explicit expression for \( \vec{h} \) in terms of \( \vec{D} \) by transcribing Eq. (60), noting that \( \vec{h} \) is the cognate of \( P_2 \) and that the cognate of \( P^1 \) is absent from the electrostatic field problem. Thus, we have

\[
h = -(\text{curl } \varepsilon^{-1} \text{ curl})^{-1} \text{ curl } \varepsilon^{-1} \vec{D},
\]

which, together with Eq. (84), yields

\[
\vec{d} = -\text{curl} (\text{curl } \varepsilon^{-1} \text{ curl})^{-1} \text{ curl } \varepsilon^{-1} \vec{D}.
\]

Equation (86) is of interest because it shows that the value of \( \vec{d} \) depends explicitly on that of \( \vec{D} \), a property whose significance will become clear shortly. (Incidentally, Eqs. (85) and (86) are quite consistent with the usual equations of electrostatics since Eq. (85) may be reduced directly to curl \( \vec{E} = 0 \) by re-inverting (curl \( \varepsilon^{-1} \text{ curl})^{-1} \) and then using Eqs. (83) and (84). Conversely, Eq. (85) could have been derived by the reverse procedure.)

The usual problem in electrostatics is to find \( \phi \) and/or \( \vec{E} \) when \( \rho \) is given, along with certain boundary conditions. By transcribing Eq. (66), we obtain the expression,

\[
\phi = -(\text{div } \varepsilon \text{ grad})^{-1} \text{ div } \vec{D}
\]

or

\[
\phi = -(\text{div } \varepsilon \text{ grad})^{-1} \rho
\]

so that, in view of Eq. (82),

\[
\vec{E} = \text{grad} (\text{div } \varepsilon \text{ grad})^{-1} \rho.
\]

Thus, while \( \rho \) alone is adequate to determine \( \phi \) and/or \( \vec{E} \) and even \( \vec{D} + \vec{d} \), the total displacement, \( \vec{D} \) itself must be given explicitly in order to determine \( \vec{d} \) and/or \( \vec{h} \). But there are an infinite number of vectors \( \vec{D} \) corresponding to the same \( \rho \) since Eq. (81) like (Eq. (16), for example) defines a many-to-one correspondence. This point is of no real consequence in electrostatics, but it is important in specifying precisely the electromagnetic field problem.

The Magnetostatic Field Problem

The operational structure pertinent to the magnetostatic field is shown in Fig. 13. Here, the variables and their dimensions are:

- \( \vec{A} \), the vector potential in webers/meter;
- \( \vec{B} \), the induction field in webers/(meter\(^2\));
- \( \vec{H} \), the magnetic field intensity in amperes/meter;
- \( \vec{i} \), the current density in amperes/(meter\(^2\)).

Again, dimensional consistency requires invoking the duality concept.

The fundamental equations for magnetostatics are

\[
\vec{i} = \text{curl } \vec{H}
\]

\[
\vec{B} = \text{curl } \vec{A}
\]
and

$$\vec{B} = \mu \vec{H}$$  \hspace{1cm} (92)

where $\mu$ is the magnetic permeability. The usual problem is to find $\vec{A}$ and/or $\vec{B}$ when $\vec{i}$ is given. By transcribing Eq. (61), observing that the cognate of $P^2$ is absent, we obtain the result

$$\vec{A} = (\text{curl} \, \mu^{-1} \, \text{curl})^{-1} \vec{i}$$  \hspace{1cm} (93)

from which it follows that

$$\vec{B} = \text{curl} \, (\text{curl} \, \mu^{-1} \, \text{curl})^{-1} \vec{i}.$$  \hspace{1cm} (94)

The magnetic field intensity $\vec{H}$ could then be obtained using Eq. (92).

Obviously, the current density vector $\vec{i}$ in Fig. 13 is solenoidal and like the vector $\vec{d}$ in Fig. 12, corresponds to the symbol $\vec{M}^2$ in Fig. 9. But $\vec{i}$ and $\vec{d}$ have different dimensions, and so cannot correspond to each other. However, the displacement current $\vec{d}$ has the same dimensions as $\vec{i}$ so that it can—and, of course, does—correspond thereto in the electrodynamic field problem.

The Electrodynamic and Magnetodynamic Fields

Figures 12 and 13 are easily extended to describe the electrodynamic and magnetodynamic fields. The only changes required are to introduce the operator $\partial/\partial t$, to insert the time derivatives $\dot{\rho}$, $\vec{D}$, $\vec{d}$, $\vec{A}$, and $\vec{B}$, and to change the signs of $\mu$, $\dot{A}$, and $\vec{B}$, as shown in Fig. 14. Once this has been done, it becomes evident not only that $\vec{i}$ and $\vec{d}$ have the same dimensions, but also that $\vec{A}$ and $\vec{E}$ are dimensionally alike and so may be merged to define the total electric field intensity.
\( \vec{E} - \vec{A} \). At the same time, \( \vec{I} \) and \( \vec{d} \) being cogenates of \( M^2 \) may not only be merged but identified. Thus, \( \vec{d} \), the solenoidal component of displacement current, becomes the source of \( \vec{H} \) according to the relation

\[
\vec{d} = \text{curl} \, \vec{H}.
\]  

(95)

This is the focal point of Maxwell's explanation of how the electric and magnetic fields are dynamically coupled.

The final step leading to the topological description of the operational structure underlying Maxwell's equations is straightforward. As shown in Fig. 15, the electro- and magnetodynamic field equations are coupled together, giving rise to a structure which makes use of most, but not all, of Fig. 11. The conductivity \( \sigma \) is now taken into account, so that the basic relation between the electric field intensity and the current density is

\[
\vec{I} + \vec{i} = \left( \sigma + \varepsilon \frac{\partial}{\partial t} \right) (\vec{E} - \vec{A}).
\]  

(96)

Here \( \vec{I} + \vec{i} \) represents the sum of both the conductive current \( \sigma (E - A) \) and the displacement current which must be defined as

\[
\vec{D} + \vec{d} = \varepsilon \frac{\partial}{\partial t} (\vec{E} - \vec{A}).
\]  

(97)

Now from Eqs. (96) and (97), it is evident that the solenoidal component of the total current \( \vec{i} \) (which is comprised of both conductive and displacement currents) is produced partly by the changing magnetic flux as indicated by the variable \( \vec{A} \). At the same time, in view of Eqs. (90) through (95), \( \vec{I} \) appears to be responsible for the existence of \( \vec{A} \), as well as \( \vec{B} \) and \( \vec{H} \). It is impossible to say, therefore, which of the two variables \( \vec{i} \) and \( \vec{A} \) is "cause" and which is "effect."
The important point to recognize is that both variables are simultaneously involved in the dynamic coupling between the electric and magnetic fields.

This point is made clear by the algebra of the 3-network problem, particularly Eqs. (64), (65), and (66), which also show that all of the electromagnetic field variables $\bar{I}$, $\bar{H}$, $B$, $A$, $E$, $\phi$, and $\rho$ are "effects" of the variable $\bar{T}$. For example, if we transcribe Eqs. (64) and (65) in terms of these variables (observing that the cognate of $P^2$ is absent) we obtain the expressions

$$\dot{\bar{A}} = -[\text{curl}(f + w) \text{curl}]^{-1} \text{curl } f \text{ curl } u\bar{T}$$

and

$$\bar{H} = -[\text{curl}(g + u) \text{curl}]^{-1} \text{curl } u\bar{T},$$

where

$$u = (\sigma + \varepsilon \partial / \partial t)^{-1},$$

$$w = (\mu \partial / \partial t)^{-1},$$

$$f = (\text{curl } u \text{ curl})^{-1},$$

$$g = (\text{curl } w \text{ curl})^{-1}. $$

Then, using Eqs. (90) and (99), we have

$$\bar{T} = -\text{curl } [\text{curl } (g + u) \text{curl}]^{-1} \text{curl } u\bar{T},$$

and by transcribing Eq. (66), we find that

$$\phi = -[\text{div}(\sigma + \varepsilon \partial / \partial t) \text{ grad}]^{-1} \text{div } [\bar{T} + (\sigma + \varepsilon \partial / \partial t) \dot{\bar{A}}].$$
Finally, using \( \mathbf{B} = \mu \mathbf{H} \), \( \mathbf{E} = -\nabla \phi \), and \( \rho = \text{div} \mathbf{I} \), we see that all of the field variables are ultimately functions of \( \mathbf{I} \). In other words, \( \mathbf{I} \) is the primary source of the electromagnetic field.

It may be helpful to think of the current density variable \( \mathbf{I} \) as playing a role similar to that of the branch current source vector \( I \) in the electrical network problem since both quantities may be arbitrarily specified. The solenoidal current density \( \mathbf{I} \), then, can be regarded as a response variable analogous to the branch current vector \( i \). The variable \( \mathbf{A} \), however, even though it does behave like a voltage source, is not completely analogous to the branch voltage source vector \( E \), since \( \mathbf{E} \) may be arbitrarily specified whereas \( \mathbf{A} \) may not. This is due to the fact that \( \mathbf{A} \) is essentially a response variable in the more intricate operational algebra of the 3-network problem.

In the foregoing discussion, we have purposely ignored the role of boundary conditions, since our main intent has been to exhibit the basic relationships between, and the topological character of, the different variables in the electromagnetic field problem. Accordingly, the operational equations derived represent only formalized "solutions" to the underlying partial differential equations, and would require the inclusion of the boundary conditions before being implemented computationally.

The use of a finite 3-network model for this implementation is, of course, a definite possibility, but the details of how to incorporate boundary conditions and carry out the computations most effectively remains to be investigated. Two points in particular that should be explored are: first, the relative size of the matrices involved and the accuracy obtainable with a 3-network model as opposed to the linear graph model of the electromagnetic field proposed by Kron[5], and second the extension of Kron's method of interconnecting solutions [24], [29], [31] to the 3-network problem.

**NETWORK MODELS FOR PARTIAL DIFFERENTIAL EQUATIONS**

On the basis of the treatment given in the previous section, we can justify the existence of network models for two important classes of partial differential equation in addition to Maxwell's equations. For example, using Eq. (88), we may write

\[
\text{div} \ \varepsilon \ \nabla \phi = -\rho. \tag{106}
\]

Since \( \rho \) is an arbitrarily specifiable quantity, we may make it a function of \( \phi \) and its time derivatives, thereby defining the generally partial differential equation

\[
\text{div} \ \varepsilon \ \nabla \phi = k + a_0 \phi + a_1 \partial \phi / \partial t + a_2 \partial^2 \phi / \partial t^2 \tag{107}
\]

where \( k, a_0, a_1, \) and \( a_2 \) are disposable parameters.

Obviously, a dual 3-network model still applies to Eq. (107). But it is also possible to use a linear graph or 1-network model instead. To show this, we simply re-invoke the duality principle and replace the dual cochain sequence of Fig. 12 with a primal chain sequence, thereby constructing the vector calculus counterpart of Figs. 3 and 4 for the linear graph. (Since the 2-cells and 3-cells are not needed, we can consider that we have, in effect, only a linear graph.)
Thus, 1-network models can properly be used to represent Eq. (107) which encompasses Laplace's, Poisson's, and Helmholtz's equations as well as the diffusion equation and the wave equation. It should be noted, incidentally, that for the latter two equations RC and RLC network models are required. The nodal method of solution applies in all cases.

Another broad class of partial differential equation which may be represented by network models may be obtained from Eq. (93) by letting \( \tau \) be a function of \( \vec{A} \) and its time derivatives. Thus, we may write

\[
\text{curl } \mu^{-1} \text{curl } \vec{A} = \vec{K} + b_0 \vec{A} + b_1 \frac{\partial \vec{A}}{\partial t} + b_2 \frac{\partial^2 \vec{A}}{\partial t^2}
\]  

(108)

where \( \vec{K} \), \( b_0 \), \( b_1 \), and \( b_2 \) are disposable parameters. Once again we re-invoke the duality principle, this time replacing the primal cochain sequence of Fig. 13 with a dual chain sequence. The resulting transformation diagram corresponds to that of a 2-network since 2-cells are directly involved. Accordingly, a 2-network model must be used for representing Eq. (108).

There are many other partial differential equations based on the vector calculus which should be amenable to this topological treatment—e.g., the equations for hydrodynamic, elastic, and magnetohydrodynamic fields. A study of these equations from a topological point of view should prove interesting and may lead to better network models for computational purposes.

**SUMMARY**

1). The algebraic-topological basis for network analogies and the vector calculus has been described.

2). It has been shown that network analogies exist for a wide variety of physical phenomena because the variables involved in these phenomena obey the same topological principles as those involved in the network problem. Ground rules for establishing the existence of network models have been given and electrical, mechanical, and structural networks have been cited as examples.

3). The network problem associated with a linear graph has been extended to a higher-dimensional topological structure, namely a simply-connected 3-complex. By invoking a duality principle, the algebraic structure underlying the 3-network problem has been modified so as to form the pattern on which an operational structure for the vector calculus can be derived.

4). The topological character of the vector calculus operators gradient, curl, and divergence and their interrelations have been described in terms of and shown to be cognates of the coboundary operators in the 3-network problem.

5). On this basis, a novel topological interpretation of Maxwell's equations has been given which shows in detail how the electrodynamic and magnetodynamic fields are coupled together.

6). Finally, the existence of network models for two important classes of partial differential equation based on the vector calculus is justified.

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APPENDIX A

The property of a linear transformation (over the complex coefficient field) which Roth [19] calls "ohmicness" is a weaker—hence, a more general—condition than that of "power definiteness" [18] which, in turn, is weaker than "positive definiteness." A transformation such as $S_1$ or $S_2$ in Fig. 7 is said to be ohmic if, for each nonzero vector $q_1$ in $Q_1$, the scalar quantity $q_1^* S_1 q_1$ is also nonzero, where $q_1^*$ is the conjugate transpose of $q_1$. That is, the transformation $S_1$ never maps a nonzero vector $q_1$ into another vector orthogonal to $q_1$.

If $S_1$ is ohmic, then its inverse $T_1$ is likewise ohmic. For $S_1 q_1 = q_1^*$ and $q_1^* = (T_1 q_1^1)^*$, so that $q_1^* S_1 q_1 = (T_1 q_1^1)^* q_1^1$. Moreover, $q_1 \neq 0$ implies $q_1^1 \neq 0$. Hence, the ohmicness of $S_1$ guarantees that

$$q_1^* S_1 q_1 = (T_1 q_1^1)^* q_1^1 \neq 0 \quad (A.1)$$

when $q_1 \neq 0$, or equivalently, when $q_1^1 \neq 0$. Accordingly, $T_1$ is ohmic. The same applies, of course, to $S_2$ and $T_2$.

Now, extending Roth's theorem to the case of a simply-connected 3-complex, we assert that if $S_1$ and $S_2$ are ohmic, then the inverse matrices $(C_{01} T_1 C_{01}^t)^{-1}$, $(C_{12} S_1 C_{12})^{-1}$, $(C_{12} T_2 C_{12}^t)^{-1}$, and $(C_{23} S_2 C_{23})^{-1}$ exist. This assertion may be proved on the basis of the general theorem that if $x = 0$ is the only solution of the matrix equation $Ax = 0$, then the inverse of $A$ exists [32].

We need to demonstrate, therefore, that the only solution of

$$(C_{01} T_1 C_{01}^t) p^0 = 0 \quad (A.2)$$

is $p^0 = 0$, the zero vector of $P^0$. This will assure the existence of the inverse, $(C_{01} T_1 C_{01}^t)^{-1}$. Assuming that Eq. (A.2) is true, it follows that

$$p^0 C_{01} T_1 C_{01}^t p^0 = 0 \quad (A.3)$$

where $p^0^*$ is the conjugate transpose of $p^0$. Since $m^1 = C_{01}^t p^0$, where $m^1$ is a vector in the subspace $M^1$ of 1-cocycles, it is clear that

$$m^1^* T_1 m^1 = 0. \quad (A.4)$$
But the ohmicness of $S_1$ and of $T_1$ requires that Eq. (A.4) can be true only if $m^1 = 0$. Therefore, it must be true that

$$C_{01}^T P^0 = 0$$  \hspace{1cm} (A.5)

Since the matrix $C_{01}^T$ is identical with the $A$ matrix for a linear graph, we may write in place of Eq. (A.5) the relation

$$A P^0 = 0$$  \hspace{1cm} (A.6)

or, partitioning $A$ as in Eq. (3) and discarding the submatrix $A_L$, we have

$$A_T P^0 = 0.$$  \hspace{1cm} (A.7)

But from Eq. (1) it follows that

$$P^0 = A_T^{-1} 0 = B_T^T 0 = 0.$$  \hspace{1cm} (A.8)

Accordingly, $P^0 = 0$ is the only solution of Eq. (A.2), proving that $(C_{01} T_1 C_{01})^{-1}$ exists.

The proof of the existence of $(C_{23}^T S_2 C_{23})^{-1}$ is quite similar. Starting with the assumption that

$$(C_{23}^T S_2 C_{23}) P_3 = 0$$  \hspace{1cm} (A.9)

and that $S_2$ is ohmic, we can easily show that of necessity

$$C_{23} P_3 = 0$$  \hspace{1cm} (A.10)

In order to show that this implies $P_3 = 0$ as the only solution to Eq. (A.9), we invoke the dual 3-complex of Fig. 8 and recall that the $C_{23}$ matrix in Eq. (A.10) is identical with the $C_{01}^T$ matrix of the dual 3-complex, this matrix, in turn, being identical with the $A$ matrix for a linear graph. We then resort to the reasoning of Eqs. (A.7) and (A.8) to conclude that $P_3 = 0$. Hence, the ohmicness of $S_2$ will insure the existence of $(C_{23}^T S_2 C_{23})^{-1}$.

To prove the existence of $(C_{12}^T S_1 C_{12})^{-1}$, we may proceed as follows. Let

$$(C_{12}^T S_1 C_{12}) q_2 = 0$$  \hspace{1cm} (A.11)

where $q_2$ is any vector belonging to the entire space $Q_2$. It follows that

$$q_2^T C_{12}^T S_1 C_{12} q_2 = 0.$$  \hspace{1cm} (A.12)

But since $m_1 = C_{12} q_2$, where $m_1$ belongs to the subspace $M_1$, we must have

$$m_1^T S_1 m_1 = 0$$  \hspace{1cm} (A.13)

while the ohmnicness of $S_1$ insures that $m_1 = 0$, so that

$$C_{12} q_2 = 0.$$  \hspace{1cm} (A.14)

Now $M_2$ is the subspace of all 2-cycles. Yet here is a vector $q_2$ which by hypothesis is not confined to $M_2$ and still has a vanishing boundary. It must follow, then, that $q_2$ can only be a zero vector. Accordingly, $q_2 = 0$ is the only solution to Eq. (A.11) and so $(C_{12}^T S_1 C_{12})^{-1}$ exists. The existence of $(C_{12} T_2 C_{12}^T)^{-1}$ may be proved in a parallel manner, the details of which need not be repeated.
An auxiliary property of some interest if the following: if $S_1$ is ohmic, then the congruence transformation $C^t_{12}S_1C_{12}$ is also ohmic. The proof is trivial, for if we choose a vector $q_2 \neq 0$, it is obvious that $m_1 = C_{12}q_2 \neq 0$. The ohmicness of $S_1$ then insures that

\[ m^*_1S_1m_1 = q^*_2(C^t_{12}S_1C_{12})q_2 \neq 0 \]  

(A.15)

from which it follows that $(C^t_{12}S_1C_{12})$ must also be ohmic.

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