Global formulation of 3D magnetostatics using flux and gauged potentials

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SUMMARY

The use of algebraic formulation for the solution of electromagnetic fields is becoming more and more widespread. This paper presents the theoretical development of two algebraic formulations of the magneto-static problem and their implementation in a three dimensional computational procedure based on an unstructured tetrahedral mesh. A complete description of the variables used and of the solution algorithm is provided together with a discussion about the performances of the method. The performances of the two procedures are tested and assessed versus cases with known solutions.

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1. INTRODUCTION

The numerical solution of electromagnetic field problems is traditionally approached by discretising Maxwell’s equations in differential form. These equations are often inserted in some variational scheme, like for instance finite element method, to obtain a discrete numerical solution. An alternative approach is based on finite difference techniques, where a direct approximation of the differential operators is performed by using incremental ratios. This approach has seen a remarkable success in high frequency electromagnetic field problems where the use of the Finite Difference Time Domain scheme [1] is now widespread in many research works and engineering applications.

A different line of reasoning has been frequently adopted in other research areas. In thermal and fluid dynamic environments, a numerical solution of the governing equations by means of control-volume technique is often performed. This technique does not take directly into account scalar or vector fields as solution variables but instead their integral values over some space-time region. In this way, the fundamental laws of the physical phenomenon, for instance

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mass balance, are imposed directly by means of algebraic equations over a control-volume etc.. As is well defined in the introduction of Patankar [2], this approach is a step behind with respect to classical formulation of the fields, returning to the *pre-calculus* days when balance equations over control-volumes would have been the only way to write down a mathematical model.

This approach is usually addressed as *finite* or *global* formulation of the physical problem. Some efficient applications of this scheme can be found also in the electromagnetic area. Besides the already mentioned FDTD method, the Finite Integration Technique proposed by Weiland is a well known example [3].

Finite formulation provides a theoretical frame which can be easily implemented in a computational procedure defined on a structured regular grid. Furthermore, the definition of intertwined grids with related physical variables reveals effective, as for instance with the use of staggered grids as proposed in [2]. Notwithstanding the enormous advantages of this implementation, both on the side of computational efficiency and of easy writing of code, its main drawback is related to the poor geometrical discretization. A regular grid, in fact, is not well suited for the treatment of curved boundaries or of uneven geometrical dimensions inside the domain of the problem. Some attempts to overcome this limitation have been devised, but they often lead to very complex solutions, see for instance [4]. In this respect, the use of the finite element method on a non regular simplicial mesh is without doubt more efficient.

In the development of electromagnetic analysis the contribution of Tonti in the definition of Global Formulation of Electromagnetic Fields (GFEF), also referred to as Finite Formulation, [5], allows a geometrical reinterpretation of the physical laws of electromagnetism and of the corresponding constitutive equations staring from the definition of the *global variables*. Global variables are equivalent to the integral variables commonly used in the classical formulation of the electromagnetism. For this reason we refer here to the Tonti’s work as *Global formulation* instead of Finite formulation.

Another concept of the Global formulation is related to the introduction of a pair of oriented cell complexes, mainly based on simplexes, one the dual of the other, where the duality present in Maxwell’s equations can be efficiently translated in a space-time discretization. In this way, the physical laws of electromagnetism become topological constraints applied on the global variables and they can be enforced also on irregular space-time cells. On the other hand, constitutive equations are relations between point-wise objects and, as it will be shown later in this paper, to be derived in a discrete form they require metric notions and an hypothesis on the field distribution within a cell.

The main advantage of the proposed approach, in authors opinion, is in the remarkable simplification of theoretical formulation which is needed deriving an algebraic system equivalent to that of the FE method. Even though the final system of equations can be proved to be the same of that derived with FE, nevertheless the approach remains still interesting because it highlights the geometric and physical aspects of electromagnetics and it allows to univocally assign the global physical variables to the suitable geometrical entities (points, lines, surfaces and volumes) endowed with orientation (inner or outer).

Because at the foundation of GFEF are the basic ideas of algebraic topology, the implementation of a solution algorithm for field problems, for many aspects, it resembles the approaches to the solution of electric networks. This theoretical treatment can be applied to the solution of the full Maxwell’s scheme involving space and time discretization. In this work, GFEF has been applied to the solution of magnetostatics, limiting thus to the space
tessellation of the domain.

Two formulations of the magneto-static problem are presented and implemented: one using magnetic flux as solution variable and the other using the circulation (line integration along an open line) of the magnetic vector potential. In both the cases, the solution algorithm requires the definition of a tree along some mesh edges in order to find out the set of independent variables and equations.

The problem of uniqueness and gauging of magnetic vector potential has been addressed by many researchers, see for instance [6], [9], [10], [11] and [12] in the frame of variational formulation. Here this problem is recast in a discrete way linking the minimal set of independent unknowns to the minimal set of variables needed to define the solution. In this way, gauge conditions are no more related to the space where the solution is looked for, but directly to the topological structure of the discretization. Similar approaches for a physical definition of gauge conditions can be found in [15], [7], [8].

The paper presents the subject with the following structure: section 2 recalls the main aspects of the global formulation particularized for the case of magnetostatics, section 3 considers the magneto-static laws recast in global form, and 4 presents the approaches to the magnetic constitutive equation. In section 5 flux and magnetic vector potential formulations are discussed respectively, while section 6 reports some results on test cases and then conclusions are drawn.

2. Global Formulation of magnetostatics

According to the Global Formulation, it is possible to deduce a set of algebraic equations directly from physics instead of obtaining them from a discretisation process applied to differential or integral equations written in terms of the field quantities. Global variables are the starting point of the global formulation; using the global variables, like currents or magnetic fluxes, a direct discrete formulation of physical laws can be obtained ready for the numerical implementation.

Global variables are referred not only to points or instants, like the field variables, but also to oriented space and time elements like points \( P \), lines \( L \), surfaces \( S \), volumes \( V \), time instants \( I \) and time intervals \( T \) (the bold face indicates that the elements are oriented) and are therefore domain functions instead of point functions. Moreover global variables are continuous in the presence of different materials and do not require any restriction, like field functions do, in terms of derivability conditions on the material media parameters.

The global variables, used in static magnetic field analysis are:

- the magnetic flux \( \Phi \);
- the circulation \( p \) of the magnetic vector potential;
- the electric current \( I \);
- the magnetic voltage \( F \).

The Global Formulation is also based on a further classification of the physical variables, into configuration, source and energy variables.

The configuration variables describe the configuration of the field or of the system without the intervention of the material parameters; also those variables linked to them by algebraic operations like sum or division by a length or an area are configuration variables. Examples of
the configuration variables are: the magnetic flux $\Phi$ or the circulation $p$ of the magnetic vector potential, as well as the induction field $B$, and the magnetic vector potential $A$.

The source variables describe the sources of the field without involving the material parameters; also those variables linked to them by algebraic operations like sum or division by a length or an area are source variables. Examples of the source variables are: the electric current $I$, the magnetic voltage $F$, as well as the electric current density $J$ or the magnetic field $H$.

The energy variables are obtained as the product of a configuration variable by a source variable; the magnetic energy density is an example of an energy variable.

The global variables involved in magnetostatics are reported in Table I. Their dependence on the oriented geometrical elements (in bold face) is evidenced within a square bracket; moreover, to distinguish one of the two possible orientations (inner and outer), a tilde is used to specify the outer orientation respect to the inner orientation, as will explained later. The global variables are related to the field functions by means of an integration performed on oriented lines, surfaces, volumes; for this reason they are equivalent to the commonly used integral variables.

| configuration global variables (Wb) | $\Phi[S] = \int_S B \cdot dS$ | $p[L] = \int_L A \cdot dL$ |
| source global variables (A) | $I[S] = \int_S J \cdot dS$ | $F[L] = \int_L H \cdot dL$ |

Table I. Global variables in magnetostatics and their relation with the field functions. These variables are also divided into configuration and source variables.

2.1. Space discretisation

The implementation of the Global Formulation requires the use of a pair of oriented cell complexes. The oriented geometrical elements, into which the space is discretized, can be thought as parts of a primal cell complex that fills the whole region where the physical phenomena are considered, Fig. 1. Vertices $p_h$, edges $l_i$, faces $s_j$ and cells $v_k$ are representative of points $P$, lines $L$, surfaces $S$ and volumes $V$ of the primal cell complex $K$; the subscripts are required in computational electromagnetism to number the geometrical elements.

From the primal complex, we can introduce the dual complex $\tilde{K}$ made of geometrical elements denoted by $\tilde{v}_h$, $\tilde{s}_i$, $\tilde{l}_j$, $\tilde{p}_k$ with a tilde to distinguish them from the corresponding geometrical elements of the primal cell complex $K$. The duality between the two complexes $K$ and $\tilde{K}$ assures that the geometrical elements correspond as follows: $p_h \leftrightarrow \tilde{v}_h$, $l_i \leftrightarrow \tilde{s}_i$, $s_j \leftrightarrow \tilde{l}_j$, $v_k \leftrightarrow \tilde{p}_k$.

The primal cell complex $K$ considered here is based on tetrahedral cells $v_k$, while the dual complex $\tilde{K}$ is built starting from $K$ according to the barycentric subdivision. A dual node $\tilde{p}_k$ is the barycentre of the tetrahedron $v_k$; a dual edge $\tilde{l}_j$, crossing a primal face $s_j$, is built in two segments each one going from the barycentre of the face $s_j$ to the barycentres of the two tetrahedra shearing that face. A dual face $\tilde{s}_i$, crossed by a primal edge $l_i$, is the union of
a number of quadrilateral facets; each of them is delimited by the following four nodes: the barycentre of the edge $l_i$, the barycentre of the tetrahedron individuated by the two faces $s_j$ having the edge $l_i$ in common, the barycentres of those two faces $s_j$. A dual cell $\tilde{v}_h$ forms a volume enclosing a primal node $p_h$; its boundary faces are a number of dual faces $s_j$.

In the paper the domain of interest is discretised into $N_v$ tetrahedra as primal cells $v_k$, in $N_s$ primal faces $s_j$, in $N_l$ primal lines $l_i$ and $N_p$ primal points $p_h$. Due to the duality between $K$ and $\tilde{K}$ it can be stated that $N_v$ is the number of dual points $\tilde{p}_k$, $N_s$ is the number of dual lines $\tilde{l}_j$, $N_l$ is the number of dual faces $\tilde{s}_i$, and $N_p$ is the number of dual cells $\tilde{v}_h$.

### 2.2. Orientation of cell complexes

Points, lines, surfaces and volumes of the two cell complexes $K$, $\tilde{K}$ are endowed respectively with inner and outer orientation, (see Fig. 1 right). A point $p_h$ can be oriented as a sink, this gives the inner orientation of it, the inner orientation of a line $l_i$ is a direction chosen on it, the inner orientation of a face $s_j$ is a direction to go along its boundary and the inner orientation of a cell $v_k$ is a congruent inner orientation of all its bounding faces (e. g. all faces taken with counter-clock wise direction).

Considering now the dual complex $\tilde{K}$, automatically all geometrical elements of $\tilde{K}$ are endowed with outer orientation. Precisely the outer orientation of a volume $\tilde{v}_h$ corresponds to an outward direction across its boundary faces and it is the inner orientation of the point $p_h$, the outer orientation of a face $\tilde{s}_i$ is the inner orientation of the primal line crossing it, the outer orientation of a dual edge $\tilde{l}_j$ is the inner orientation of the primal face $s_j$ crossing the it, the outer orientation of a dual point $\tilde{p}_k$ is the inner orientation of the primal cell $v_k$ containing the point.

### 2.3. Global variables and cell complexes

The classification of global variables into configuration and source variables has a paramount impact in the numerical implementation of the Global Formulation. It has been shown that configuration variables are naturally associated with geometrical elements endowed with inner orientation and therefore are referred to the primal complex $K$; source variables are associated with geometrical elements with outer orientation and are therefore referred to the dual complex.
This unique association of the global variables to the respective primal or dual cell complex, is one of the key points of the Global Formulation. Therefore the global variables of magnetostatics are associated to the oriented geometrical elements of the cell complexes $K, \tilde{K}$ as follows:

- $\Phi$ is the vector, of dimension $N_s$, of the fluxes relative to the primal faces $s_j$;
- $p$ is the vector, of dimension $N_l$, of the circulations of $A$ relative to the primal edges $l_i$;
- $I$ is the vector, of dimension $N_l$, of the electric currents across the dual faces $\tilde{s}_i$;
- $F$ is the vector, of dimension $N_s$, of the magnetic voltages relative to the dual edges $\tilde{l}_j$.

3. Global form of magneto-static laws

The physical laws of magneto-static, rewritten in *global form*, are expressed as topological constraints on global variables of the same kind. These constraints tie a global variable associated to a geometrical element to be equal to another global variable associated to its boundary. With reference to the pair of primal-dual cell complexes $K, \tilde{K}$ above defined, the physical laws governing the magneto-static analysis are:

- the *magnetic Gauss* law:
  \[ D\Phi = 0 \]  
  with $D$ the incidence matrix, of dimension $N_v \times N_s$, between the inner orientations of a primal cell $v_k$ and a primal face $s_j$; $\Phi$ is the vector of fluxes;
- the *Ampère* law:
  \[ \tilde{C}F = I \]  
  with $\tilde{C}$ the incidence matrix, of dimension $N_l \times N_s$, between the outer orientations of a dual face $\tilde{s}_i$ and a dual edge $\tilde{l}_j$; $F$ and $I$ are the vectors of magnetic voltages and currents respectively;
- the *charge conservation* law:
  \[ \tilde{D}I = 0 \]  
  with $\tilde{D}$ the incidence matrix, of dimension $N_p \times N_l$, between the outer orientations of a dual cell $\tilde{v}_h$ and a dual face $\tilde{s}_i$; $I$ is the vector of currents.

4. Magnetic constitutive equation

The link between configuration and source variables are the *constitutive equations* that contain the material properties and the metrical notions such as lengths, areas and volumes. Aim of this section is to derive a finite dimensional linear operator, i.e. a matrix, linking the magnetic voltages relative to dual edges to the fluxes relative to primal faces. In the framework of differential forms, this linear operator is a discrete approximation of the *Hodge* operator linking the 1-(twisted) form of magnetic voltages to the 2-form of fluxes; a contribution around this issue, is given by the papers [13] and [14]. In the case of magneto-static analysis, the *magnetic constitutive* equation can be derived according to two equivalent approaches described in the following. In both of the approaches the media inside a primal cell $v_k$ is assumed as uniform.
with permeability matrix $\mu^k$ of dimension 3; moreover it is convenient to introduce the local vectors of fluxes $\Phi^k = [\Phi_1, ..., \Phi_4]^T$ and of the magnetic voltages $F^k = [F_1, ..., F_4]^T$ along the four portions of dual edges $I_1, ..., I_4$ inside the tetrahedron and with local numbering. It should be noted that, due to the barycentric subdivision, the primal face $s_j$ is not orthogonal to the portion of dual edge $\tilde{I}_j$, Fig. 2.

**Figure 2.** A detail of a primal cell $v_k$ (a tetrahedron) is displayed together with the four primal nodes $a, b, c, d$, the six primal edges $l_1, ..., l_6$ and the four primal faces $s_1, ..., s_4$ with local numbering. Moreover the dual node $\tilde{p}_k$ located at the barycentre of the tetrahedron is shown together with the four portions of dual edges $\tilde{I}_1, ..., \tilde{I}_4$, with local numbering, and a portion of a dual face $\tilde{s}_i$ tailored inside the primal cell.

### 4.1. Facet elements interpolation

In the first approach an interpolation is performed to locally link the two vectors $F^k, \Phi^k$. This interpolation is built using Whitney facet element shape functions inside each tetrahedron, [17]. Using these interpolating functions, the magnetic flux density in each primal cell can be written as:

$$B^k = \sum_{j=1}^{4} d_{kj} w_j \Phi_j$$  \hspace{1cm} (4)

being $d_{kj}$ the incidence numbers between the inner orientations of the primal cell $v_k$ and of its primal faces $s_j$, $w_j$ are the facet elements shape function vectors for the $j$-th face having nodes $(l, m, n)$ as vertices; it is defined as:

$$w_k = 2 \left[ N_l(P) \nabla N_m \times \nabla N_n + N_m(P) \nabla N_n \times \nabla N_l + N_n(P) \nabla N_l \times \nabla N_m \right]$$  \hspace{1cm} (5)

where $N_l(P)$ is the affine nodal shape function for the node $l$. From (4), it can be easily verified that the induction $B^k$ is uniform within a tetrahedron and therefore a uniform field $H^k$ can be deduced from the constitutive law in terms of the fields quantities:
\[ H^k = (\mu^k)^{-1}B^k \] (6)

\( H^k \) being uniform inside \( v_k \), the magnetic voltage vector \( F^k \) can be derived as:

\[ F^k = \tilde{L}^k H^k \] (7)

where \( \tilde{L}^k \) is the \( 4 \times 3 \) matrix whose rows are the four row vectors \( \tilde{I}_1, ..., \tilde{I}_4 \) of the four portions of dual edges internal to \( v_k \).

Substituting in (7), the expressions (6) and (4), the final matrix form of the magnetic constitutive equation can be derived as:

\[ F^k = M^k \Phi^k \] (8)

\( M^k \) being the square non symmetric elemental matrix of dimension 4 relative to the cell \( v_k \).

### 4.2. Uniform field approach

A second approach to derive the local constitutive equation, starts from the assumption of uniformity of the fields \( B^k \) and \( H^k \) within the cell \( v_k \). With reference to Fig. 2, the following four vectors of the magnetic voltages along three of the four portions of dual edges can be introduced:


the field \( H^k \) can be expressed as:

\[ H^k = \tilde{L}_a^{-1}F_a = \tilde{L}_b^{-1}F_b = \tilde{L}_c^{-1}F_c = \tilde{L}_d^{-1}F_d \] (10)

where \( \tilde{L}_a, \tilde{L}_b, \tilde{L}_c, \tilde{L}_d \) are the \( 3 \times 3 \) matrices storing by rows the corresponding row vectors of the dual edges. In a compact form (10) can be rewritten in terms of the vector \( F^k \) as:

\[ H^k = A_a F^k = A_b F^k = A_c F^k = A_d F^k \] (11)

being:

- \( A_a \) a \( 3 \times 4 \) matrix whose elements are those of \( \tilde{L}_a^{-1} \) and in addition the 1-st column of zeros;
- \( A_b \) a \( 3 \times 4 \) matrix whose elements are those of \( \tilde{L}_b^{-1} \) and in addition the 2-nd column of zeros;
- \( A_c \) a \( 3 \times 4 \) matrix whose elements are those of \( \tilde{L}_c^{-1} \) and in addition the 3-rd column of zeros;
- \( A_d \) a \( 3 \times 4 \) matrix whose elements are those of \( \tilde{L}_d^{-1} \) and in addition the 4-th column of zeros.

From (11) and the inverse of the constitutive law (6), the uniform induction \( B^k \) can be derived by averaging in turn as:

\[ B^k = \frac{1}{3} \mu^k (A_b + A_c + A_d) F = \frac{1}{3} \mu^k (A_a + A_c + A_d) F = \frac{1}{3} \mu^k (A_a + A_b + A_d) F = \frac{1}{3} \mu^k (A_a + A_b + A_c) F \] (12)

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The magnetic constitutive equation expressing $\Phi^k$ as a function of $F^k$ is then derived as:

$$
\Phi^k = \frac{1}{3} \begin{bmatrix}
S_1 \mu^k (A_b + A_c + A_d) \\
S_2 \mu^k (A_a + A_c + A_d) \\
S_3 \mu^k (A_a + A_b + A_d) \\
S_4 \mu^k (A_a + A_b + A_c)
\end{bmatrix} F^k
$$

with $S_1, ..., S_4$ are the four row area vectors of the four oriented primal faces of the cell $v_k$.

Finally, by inversion of the $4 \times 4$ matrix $N^k$, exactly the magnetic constitutive matrix $M^k$ in (8) is obtained.

By assembling element by element the local constitutive equation (8), the global constitutive equation can be derived as:

$$
F = M\Phi
$$

being $M$ the $N_s \times N_s$ global constitutive matrix.

5. Formulations for magnetostatics

The algebraic system of equations that allows the solution of the magneto-static problem for given external current and media permeability distributions, can be cast according to two formulations:

- the flux formulation, where the flux vector $\Phi = [\Phi_1, ..., \Phi_{N_s}]^T$ is considered;
- the $p$—formulation, where the vector $p = [p_1, ..., p_{N_l}]^T$ of the circulations of $A$ is considered.

5.1. Flux formulation

In this case the $N_s$ fluxes relative to the primal faces $s_j$ can be split into $N_{se}$ fluxes relative to the external boundary faces and $N_{si}$ fluxes relative to the internal faces such that $N_s = N_{se} + N_{si}$. We consider in this subsection only the case of a closed boundary surface $S$, enclosing the volume $V$, on which the external fluxes $\Phi_{ej}$, with $j = 1, ..., N_{se}$, are known being assigned as boundary conditions that must comply with the magnetic Gauss law as follows:

$$
\sum_{j=1}^{N_{se}} d_j \Phi_j = 0
$$

$d_j$ are the incidence numbers between the inner orientation of the external boundary faces and the inner orientation of the external volume $V$ containing the primal cell complex $K$. The case of symmetry planes on which symmetry conditions are imposed will be treated later.

The relation (15) must be considered in addition to the $N_v$ equations from (1); therefore $N_v + 1$ equations can be written according to the Gauss law but only $N_v$ of them are linearly independent. An independent set of $N_v$ equations based on the Gauss law can be written by adding to (15), $N_v - 1$ equations obtained by eliminating from (1) a row in the matrix $D$. The corresponding reduced system becomes:

$$
D_v \Phi = 0
$$
with $D_r$ the reduced incidence matrix of dimension $(N_v - 1) \times N_s$.

By substituting the constitutive equation (14) in the Ampère's law (2), the following $N_l$ equations can be derived:

$$\tilde{C}M \Phi = I \quad (17)$$

These equations are linearly dependent due to the $N_p - 1$ independent constraints on the currents imposed by the charge conservation law (3). From the duality of the cell complexes $K, \tilde{K}$ the following relation holds:

$$\tilde{D} = -G^T \quad (18)$$

$G$ being the incidence matrix, of dimension $N_l \times N_p$, between the inner orientations of a primal line $l_i$ and of a primal point $p_h$; the rank of $G$ is $N_p - 1$ the sum of its columns being null.

The total number of independent equations becomes:

$$N_{eq} = [N_v]_{Gauss \ law} + [N_l - (N_p - 1)]_{Ampere+constitutive \ laws} \quad (19)$$

Substituting in (19) for $N_v$ the Euler's formula:

$$N_v = N_p - N_l + N_s - 1 \quad (20)$$

we obtain that the number of independent equations $N_{eq} = N_s$ equals the total number of fluxes.

Of course, from a computational point of view only $N_{si}$ fluxes relative to the internal primal faces are the actual unknowns. The corresponding set of $N_{si}$ independent equations can be derived by considering:

- $N_v - 1$ independent equations from (16), assuming (15) satisfied by the boundary conditions on the external fluxes;
- a number of closed paths can be defined, as in basic circuit theory, made of dual edges laying inside the domain. The set of internal dual edges $\tilde{l}_j$ connecting dual nodes $\tilde{p}_k$ can be treated as a connected graph $\tilde{G}$ of $N_{si}$ dual edges and $N_v$ dual points. From the graph $\tilde{G}$ a tree can be derived of $N_v - 1$ branches and the corresponding co-tree identifies the $N_{si} - N_v + 1$ fundamental loops, with respect to which $N_{si} - N_v + 1$ independent equations of the kind (17) can be written.

5.2. $p-$formulation

In this case it is convenient to introduce the vector $p$ of circulations of the magnetic vector potential, relative to the primal edges $l_i$, such that:

$$Cp = \Phi \quad (21)$$

$C$ being the incidence matrix, of dimension $N_s \times N_l$, between the inner orientations of a primal face $s_j$ and of a primal line $l_i$. Expression (21) identically satisfies the Gauss law (1), because of the identity:

$$DC \equiv 0 \quad (22)$$

Therefore by substituting (21) in (17), the following set of equations can be derived:

$$\tilde{C}M C p = I \quad (23)$$
where from the duality of the cell complexes $\tilde{C} = C^\dagger$. It can be numerically verified (and formally proved) that the $N_t \times N_t$ matrix $C^\dagger MC$ of the system (23) is symmetric even though the constitutive matrix $M$ is not.

These $N_t$ equations are not independent because of charge conservation equation (3) that imposes $N_p - 1$ independent constraints on the currents. Therefore the resulting number of independent equations is:

$$N_{eq} = N_t - N_p + 1$$  \hspace{1cm} (24)

These $N_{eq}$ independent equations constrain $N_t - N_p + 1$ circulations $p_i$; this means that the remaining $N_p - 1$ are free.

From a computational point of view it is important to identify which are the actual unknowns $p_i$ considering that some of them are imposed as boundary conditions on the external fluxes, and some of them are irrelevant in the numerical solution of (23).

By distinguishing the primal edges $N_t = N_{te} + N_{ti}$ in $N_{te}$ external boundary edges and $N_{ti}$ internal edges and the primal nodes $N_p = N_{pe} + N_{pi}$ in $N_{pe}$ external boundary nodes and $N_{pi}$ internal nodes, the number of $N_t - N_p + 1$ constrained circulations can be rewritten as:

$$N_t - N_p + 1 = [N_{te} - (N_{pe} - 1)] + [N_{ti} - N_{pi}]$$  \hspace{1cm} (25)

that can be interpreted as follows:

- considering the connected graph $G_e$ based on $N_{pe}$ boundary primal nodes and on $N_{te}$ boundary primal edges located on the external boundary, $(N_{pe} - 1)$ corresponds to the number of branches of a tree $T_e$ of the graph $G_e$;
- $N_{te} - (N_{pe} - 1)$ is the number of branches of the co-tree $C_e$ of the graph $G_e$ corresponding to the selected tree $T_e$;
- $N_{pi} - 1$ is the number of the branches of a tree $T_i$ of the graph formed by the $N_{ti}$ internal primal nodes and by a number of internal primal edges connecting those nodes; by adding to the branches of $T_i$ a further primal edge, linking an internal primal point to a primal point on the external boundary, a new graph $T'_i$ is obtained with $N_{pi}$ branches;
- the complementary graph $C'_i$ of $T'_i$ with respect to the set of the $N_{ti}$ internal primal edges can be introduced; $C'_i$ has $N_{ti} - N_{pi}$ branches.

Therefore the actual unknowns of the magneto-static problem are the $N_{ti} - N_{pi}$ values of the circulations $p_i$ associated with the branches of the graph $C'_i$, while the values of $p_i$ associated with the branches of the graph $T'_i$ can be eliminated.

Observing that the union of the graphs $T'_i$ and $T_e$ is a tree $T$ of the whole graph $G$ based on $N_p$ primal nodes and $N_t$ primal edges and that the union of the graphs $C'_i$ and $C_e$ is the co-tree corresponding to the tree $T$ of $G$, this process to identify the independent circulations $p$ is analogous the tree, co-tree decomposition of the vector potential described in [9].

5.3. Comparison of the two formulations

In the $p$–formulation the $N_v - 1$ independent equations from (16), assuming (15) satisfied by the boundary conditions on the external fluxes, are automatically satisfied thanks to (21).

The $N_{ti} - N_{pi}$ independent equations/unknowns in the system (23) are strictly related to the number of $N_{si} - N_v + 1$ independent equations of the kind (17) in the flux formulation. The Euler’s formula (20) can be rewritten as:

$$N_{pe} + N_{pi} - N_{te} - N_{ti} + N_{sc} + N_{si} - N_v = 1$$  \hspace{1cm} (26)
Using Euler’s formula (20), written for the whole volume \( V \) encasing the domain of the magneto-static problem, gives:

\[
N_{pe} - N_{le} + N_{se} = 2, \tag{27}
\]

(26) becomes:

\[
N_{si} - N_v + 1 = N_{li} - N_{pi} \tag{28}
\]

that expresses the equivalence between the two formulations.

Moreover a further correspondence between flux and \( p \)-formulations with the gauge on the magnetic vector potential suggested by Albanese-Rubinacci can be derived.

Considering the whole graph \( G \), from its internal co-tree \( C'_i \), which branches are \( N_{li} - N_{pi} \) internal primal edges, \( N_{li} - N_{pi} \) fundamental loops \( L_m \) can be constructed. \( N_{li} - N_{pi} \) independent fluxes \( \Phi^f_m \) can be associated to the fundamental loops \( L_m \), in order to form a base to express the \( N_{si} \) unknown internal fluxes \( \Phi^i_j \) associated with the corresponding internal primal faces \( s_j \), Fig. 3. Therefore the \( N_{li} - N_{pi} \) fundamental fluxes \( \Phi^f_m \) uniquely correspond to the circulations \( p^c_m \) relative to the primal edges of the internal co-tree \( C'_i \). If the circulations relative to the internal tree \( T'_i \) are set to zero the following equivalence holds:

\[
\Phi^f_m \equiv p^c_m \tag{29}
\]

assuming that each fundamental loop \( L_m \) is oriented internally as the corresponding co-tree edge.

The zeroing of the circulations relative to the internal tree \( T'_i \) corresponds to the gauge on the vector potential of Albanese-Rubinacci.

![Diagram](image_url)

Figure 3. With reference to a plane primal cellisation, a tree, the co-tree and the some of the corresponding fundamental loops are shown together with the associated independent fluxes \( \Phi^f_j \).

However if the flux formulation has a direct link with physics the resulting system obtained from (16) and (17) leads to a non symmetric matrix. Moreover, the corresponding computational scheme requires the fundamental loop topology to be stored for each dual co-tree branch. This fact in large three dimensional problems can be very demanding in terms of memory requirements. In addition, with usual tetrahedral mesh generators the obtained algebraic system is larger for the flux formulation than for the \( p \)-formulation. A further advantage of the \( p \)-formulation is that it can be easily extended to the case of magneto quasi-static problems.
5.4. Symmetry conditions

In order to reduce the computational effort the symmetry conditions can be profitably used when applicable to the magnetostatic problem. Two types of symmetry conditions can be imposed depending on the kind of the problem:

- the flux is zero relative to surfaces $\Sigma_t$, internal to the domain of interest $\mathbf{V}$; this condition is equivalent to $(\mathbf{B} \cdot \mathbf{n})_{\Sigma_t} = 0$ that expresses the tangent field condition to $\Sigma_t$;
- the flux is maximum relative to surfaces $\Sigma_n$, internal to the domain of interest $\mathbf{V}$; this condition is equivalent to $(\mathbf{H} \cdot \mathbf{t})_{\Sigma_n} = 0$ that expresses the normal field condition to $\Sigma_n$.

5.4.1. Tangent field condition

In this case the fluxes relative to the primal faces lying on the symmetry surface $\Sigma_t$ are null, together with the circulations $p$ along the primal edges bounding that faces. In this way only one fraction of the domain $\mathbf{V}$ needs to be analyzed, bounded by a part of the external boundary $\mathbf{S}$ and by the symmetry surface $\Sigma_t$. From a computational point of view this condition is treated in the same way as the boundary conditions on the external boundary $\mathbf{S}$.

5.4.2. Normal field condition

In this case the field is orthogonal to the symmetry surface $\Sigma_n$ but the fluxes relative to the primal faces on $\Sigma_n$ are unknowns as well as the circulations $p$ along the primal edges bounding that faces. In this way only one half of the domain $\mathbf{V}$ needs to be analyzed, bounded by half of the external boundary surface $\mathbf{S}$ and by the symmetry surface $\Sigma_n$. From a computational point of view the primal faces or the primal edges on $\Sigma_n$ are treated in the same way as the internal faces or the internal edges with unknown flux or circulation $p$.

The equivalence of the flux and $p$ formulations even in this case can be shown starting from (26) written as:

$$N_{pe} + N_{p\Sigma_n} + N_{pi} - N_{le} - N_{i\Sigma_n} - N_{li} + N_{se} + N_{s\Sigma_n} + N_{si} - N_v = 1 \quad (30)$$

where the geometrical entities on $\Sigma_n$ are accounted for explicitly. Because the portion of $\mathbf{S}$, where the fluxes or the circulations $p$ are assigned as boundary conditions, is an open surface and does not bound a volume anymore, the Euler’s formula gives:

$$N_{pe} - N_{le} + N_{se} = 1 \quad (31)$$

and therefore (30) simplifies as:

$$(N_{si} + N_{s\Sigma_n}) - N_v = (N_{li} + N_{i\Sigma_n}) - (N_{pi} + N_{p\Sigma_n}) \quad (32)$$

that is the corresponding relation to (28).

6. Numerical results

6.1. Implementation

The theoretical methods presented in the previous sections, have been implemented in two numerical procedures for magneto-static computations. One procedure (see example in subsection 6.1.3) is implemented in the MATLAB environment and translates the matrix
formulation of the problem in a computer code. In this case, due to the MATLAB programming environment, the number of elements must be kept as small as possible to reduce computation time and to avoid memory limits. An interface to some commercial three-dimensional mesh generators is defined and then, starting from the primal tetrahedral mesh, all the topological and material dependent matrices are computed. Another procedure has been implemented in the computationally more efficient standard programming language (Fortran). In this case, the primal mesh is generated by some external pre-processor both commercial and in-house developed. Again, starting from the tetrahedral mesh, all other geometrical entities are defined and all their topological links are stored in tables.

In the solver phase the system matrix is not assembled by means of the topological matrices but instead working on a element-by-element procedure. For each tetrahedron the local reluctivity matrix is built and then the contribution of all single entities is summed in the solution matrix. In this way the memory requirement and computational effort are minimized.

6.1.1. Cube with uniform applied field A cubic domain with a unit length edge is used to test the accuracy of the method by applying a uniform field along the \( z \) direction. A primal mesh with distorted tetrahedra made of 384 volumes, 864 faces, 604 edges and 125 nodes is used as a test bench. The domain of the problem together with the mesh are shown in Fig. 4. Boundary conditions requiring a magnetic flux density of 1 T along the \( z \) direction are applied. The solution obtained with both \( \text{flux} \) and \( p \) formulations gives an internal flux density equal to 1 T uniform at most of \( 10^{-11} \) T. The same domain has been used to assess the stability of the solution to the choice of the tree. Two trees along the mesh have been used and they have been generated by means of a \textit{depth first} and \textit{breadth first} algorithm [18]. The solution has not shown an appreciable dependence on tree choice.

6.1.2. Ferromagnetic sphere in uniform field This problem with two materials has an analytical solution which can be used as reference. The domain of the problem is shown in Fig. 5 together with the tetrahedral primal mesh. The primal mesh is made of 3240 tetrahedra, 6676 faces, 4088 edges and 653 nodes. A uniform flux is imposed on the two bases of the external cylinder corresponding to a uniform magnetic flux density of 1 T, while the sphere has a value

![Figure 4. Flux density plot produced by imposing as boundary conditions a flux density of 1 T on the upper and lower faces of the cube.](image)
of relative permeability of 1000. The vector plot of the magnetic flux density around the sphere is reported in Fig. 6. The analytical solution for the magnetic flux density inside the sphere is given by:

\[
B = \frac{3\mu_r - 1}{\mu_r + 2} B_0 = 2.9917 \, T
\]  

(33)

The results obtained by the two schemes are reported in Table II. In this case, the pattern of the solution is following the expected behaviour leading to a local increase of magnetic flux density values around the sphere and to a uniform flux density inside it. The error with respect to the analytical solution is small and can be attributed both to a discretization error of the spherical surface and to a uniform flux condition imposed at a finite distance from the sphere and not to infinity.

Table II. Comparisons for the ferromagnetic sphere case.

<table>
<thead>
<tr>
<th>flux formulation</th>
<th>p formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>B in the sphere centre [T]</td>
<td>2.853</td>
</tr>
<tr>
<td>error with respect to the analytical solution [%]</td>
<td>4.6</td>
</tr>
</tbody>
</table>

6.1.3. **Coil around a ferromagnetic cylinder** In this case a ferromagnetic cylinder with relative permeability \( \mu_r = 1000 \) is partially inserted within a coil fed with a uniform current of 1.2 A. The number of tetrahedra of the primal mesh has been kept relatively small \( N_v = 1981 \), Fig. 6 to reduce the computation time in the MATLAB solver. In Table III the amplitude of the computed flux density is reported along a number of points close to the axis of the cylindrical core compared with those obtained from a 2D axisymmetric finite elements analysis performed on a fine mesh, with a commercial code.
Figure 6. Flux density plot produced by a coil with uniform current density, around a magnetic core with constant relative permeability of $\mu_r = 1000$, only one quarter of the domain is meshed due to symmetry reasons; cylinder radius $r_c = 5\, \text{mm}$ and height $h_c = 10\, \text{mm}$, coil inner radius $R_i = 6\, \text{mm}$, outer radius $R_o = 9\, \text{mm}$. The core is $4.5\, \text{mm}$ outside the coil, while the uniform coil current is $1.2\, \text{A}$.

<table>
<thead>
<tr>
<th>Point coordinates (mm)</th>
<th>$B$ (mT) $p$ formulation</th>
<th>$B$ (mT) Finite Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.63, -0.48, 1.1)</td>
<td>0.115</td>
<td>0.122</td>
</tr>
<tr>
<td>(1.2, -0.04, 2.5)</td>
<td>0.183</td>
<td>0.193</td>
</tr>
<tr>
<td>(1.2, 0.09, 4.3)</td>
<td>0.201</td>
<td>0.259</td>
</tr>
<tr>
<td>(1.2, 0.07, 7.5)</td>
<td>0.220</td>
<td>0.271</td>
</tr>
<tr>
<td>(0.66, 0.07, 8.7)</td>
<td>0.197</td>
<td>0.217</td>
</tr>
<tr>
<td>(1.3, 0.09, 10)</td>
<td>0.161</td>
<td>0.144</td>
</tr>
</tbody>
</table>

Table III. Comparisons of the induction amplitude in the magnetic core.

7. Conclusions

The Global formulation of electromagnetic fields proposed by Tonti has been the starting point for the definition of two algorithms aiming to the solution of three-dimensional magneto-static problems. Instead of starting from a purely mathematical point of view, neglecting the nature of the underlying phenomenon, global formulation enhances geometrical and physical aspects of the electromagnetic laws. This aspect, independently from new possible achievements in numerical performances with respect to classical methods, makes global formulation important from a conceptual viewpoint.

The work presented has allowed the definition of the theoretical details of these algorithms and the assessment of their performance in a computational procedure. It can be said that:
the work has shown the feasibility of a solution based on an algebraic formulation of the electromagnetic field on an unstructured grid. This approach overcomes the difficulties related to the orthogonal and structured grids usually employed in other finite formulations;

• the flux-formulation has been based on an algorithm which is very much related to the solution of a circuit approach showing the analogies between global formulation and electric circuit;

• the p-formulation has the advantage of directly enforcing some physical properties of the flux density but introduces some problems related to the uniqueness of the solution. In this case, uniqueness is enforced using a tree-co-tree scheme and this result, which is formally identical to the one obtained by other Authors on variational basis, is obtained by enforcing physical laws;

• the study has highlighted a substantial independence of the solution accuracy on tree selection;

• the proposed formulations can have, in the Authors’s view, a notable importance for educational purposes, where a complete theoretical consistent solution scheme can be defined without requiring a deep knowledge of variational calculus.

While this study has allowed to show the feasibility of the approach, the work will have to go on to compare its computational efficiency with other well established numerical procedures based on finite elements.

8. Acknowlegments

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