Geometric Interpretation of Discrete Approaches to Solving Magnetostatics

F. Trevisan and L. Kettunen

Abstract— The finite element approach can be considered as a tool for constructing finite dimensional systems of equations that approximate to field problems on the discrete level. Although the finite element technique is explained typically in terms of variational or weighted-residual approaches, another, less familiar way is available to introduce the same ideas geometrically. Using magnetostatics as an example, we will show how finite element type system matrices can be developed by exploiting the geometric properties of so-called Whitney elements. The simple interpretation we gain thereby enables us to view also the convergence properties of finite element type schemes in an intuitive way.

Index Terms—Finite element method, cell method, differential geometry, dual meshes, convergence.

I. INTRODUCTION

The Finite Element technique, exploiting the Galerkin or the Ritz method, is a tool for creating linear systems of equations that approximate to field problems. Although the finite elements technique is the most commonly used numerical technique, alternatives do exist. Particularly, in computational electromagnetism, geometrical approaches, such as the so-called Cell Method [1] and the Finite Integration Technique [2] are recognized as very useful approaches to solving field and wave problems.

The central question that distinguishes numerical approaches from each other is how to treat the constitutive laws, or more precisely, how to construct the so-called discrete Hodge operator (see [3], [4] and [5]). The key here is that the Hodge operator, embedded in the constitutive laws, implies the metric of the problem. On the other hand, the field equations themselves, in our case Maxwell’s equations, are independent of the metric [6], [7]. In other words, the Maxwell equations themselves can be imposed exactly on the discrete level, because the approximative nature of their numerical solution follows from the constitutive laws.

This paper aims to show that the two worlds of finite elements and geometrical techniques are in fact akin to each other as suggested in [8]. We will use magnetostatics as a working example even though the use of a geometrical approach is of more general nature. We will then compare the properties of the system matrix thus obtained to matrices yielded by finite elements. Finally, we will show that both approaches converge towards the same limit.

II. PRELIMINARIES

We consider here a primal simplicial cell complex $\mathcal{K} = \{\mathcal{N}, \mathcal{E}, \mathcal{F}, \mathcal{V}\}$, whose geometrical elements are nodes $n \in \mathcal{N}$, edges $e \in \mathcal{E}$, faces $f \in \mathcal{F}$, and volumes $v \in \mathcal{V}$, all endowed with an inner orientation, [9], [10], Figure 1. As our aim is to compare the finite element method and geometrical techniques, from the primal cell complex $\mathcal{K}$, we can derive a barycentric dual complex $\hat{\mathcal{K}} = \{\hat{\mathcal{V}}, \hat{\mathcal{F}}, \hat{\mathcal{E}}, \hat{\mathcal{N}}\}$, whose geometrical elements are dual nodes $\hat{v} \in \hat{\mathcal{V}}$, dual faces $\hat{f} \in \hat{\mathcal{F}}$, dual faces $\hat{e} \in \hat{\mathcal{E}}$, and dual volumes $\hat{n} \in \hat{\mathcal{N}}$. In addition, we can exploit the inner orientation of $\hat{\mathcal{K}}$ to induce an outer orientation of $\hat{\mathcal{K}}$, provided the cells of $\mathcal{K}$ are one-to-one with those of $\hat{\mathcal{K}}$. For example, in the case of a face $f$ and its crossing dual edge $\hat{f}$, the corresponding vectors (in the following shown in roman type) $f$ and $\hat{f}$ cross $f$ in the same way. According to barycentric subdivision, a dual node $\hat{v}$ is the barycenter of a tetrahedron and a dual edge $\hat{f}$ is a broken line joining the barycenters of the two tetrahedra that share the primal face $f$ and pass through the barycenter of the face $f$. If $f$ happens to be part of the domain’s boundary, the dual edge $\hat{f}$ links the barycenters of $v$ and $f$. The mutual interconnections of the primal cell complex $\mathcal{K}$ are described by the incidence matrices: $G$ between edges $e$ and nodes $n$, $R$ between faces $f$ and edges $e$, and $D$ between volumes $v$ and faces $f$. The matrices $D^T$, $R^T$, and $-G^T$ (the minus sign comes from the assumption that $n$ is oriented as a sink, whereas the boundary of $\hat{v}$ is oriented by the outer normal) describe the mutual interconnections of $\hat{\mathcal{K}}$. The two complexes form the mesh $\mathcal{M}$.

The problem of describing the field distribution from a given mesh $\mathcal{M}$ and arrays of degrees of freedom, based on a pair of cell complexes $\mathcal{K}$ and $\hat{\mathcal{K}}$, can be recast as an idealization of a mathematical process of taking measurements. This process consists of two components, probes and a numerical reading. The mathematical model of probes, borrowed from algebraic topology, is the chain. With reference to a cell complex $\mathcal{K}$ or $\hat{\mathcal{K}}$, a $p$–chain is a formal sum of $p$–cells with $p = 0, \ldots, 3$ (0 for nodes, 1 for edges, 2 for faces, 3 for volumes) each with a relative number $c^p$. For example, a 2-chain can be expressed by the formal sum $\sum_{f \in \mathcal{F}} c^f f$ with integer coefficients $c^f$.

The second part of the process consists of obtaining the numbers to represent fluxes, currents etc. that are used to describe the electromagnetic variables. Several of these variables can be associated with a $p$–chain. This linear mapping from $p$–chains to numbers is a $p$–cochain. In this sense, integrals of fields can be understood as cochains. For instance, if we consider a surface $S$, the flux of the field $b$ relative to $S$ is $\int_S b \cdot ds$. An approximation of the flux is obtained by representing $S$ with a 2–chain $\sum_{f \in \mathcal{F}} c^f f$ such that the flux...
is $\sum_f b_f e_f$, with $b_f$ being the flux relative the face $f$ of the chain. In the following, the quantities such as $b_f$ and the array they form are denoted in boldface: $b = \{b_f : f \in F\}$ is an array of fluxes. Similarly, $h = \{h_f : f \in F\}$ is an array of magnetic voltages $h_f$ (measured in Ampère) along the dual edges $\tilde{f}$.

We now focus on a discrete model of a magnetostatic problem, which consists of the computation of arrays $b$, $h$, which are referred to a pair of cell complexes $K$, $\bar{K}$ such that

$$Db = 0 \quad (\text{Gauss law}), \quad h = Jb, \quad R^T h = j \quad (\text{Ampère's law})$$

hold, and where $j = \{j_e : e \in E\}$ is the vector of current intensities through the dual faces $\tilde{e}$ (edge-based) of a given current density $j$ such that $j = \int_{\tilde{e}} j \cdot ds$ and $G^T j = 0$ are satisfied. The constitutive matrix $\nu$ is some well chosen square matrix (of an order equal to the number $F$ of the primal faces), not necessarily symmetric. We recall that the Gauss and Ampere laws in (1) need no metric notions. These laws are valid exactly both in the large and the small in whatever media. On the other hand, in the discrete constitutive equation, the matrix $\nu$ is mesh-dependent and requires metric notions and the media characteristic to be computed.

**III. GEOMETRIC DERIVATION OF SYSTEM MATRICES**

On the discrete level, the magnetostatic problem is determined by Gauss and Ampère laws and by the constitutive law. This section aims to derive a constitutive matrix $\nu$, starting from a geometric interpretation of the Whitney basis functions of finite elements. Without losing generality, we may limit the primal mesh to a single tetrahedron $v$ under the assumption of homogeneity of the medium in it so that the reluctivity $\nu$ is constant. All results derived in this particular case can easily be extended to a mesh consisting of tetrahedra, where each element may model different media. By linearity, the global system matrix can be assembled from the contributions of single elements.

![Fig. 1. Mesh $\mathcal{M}$ limited to a single tetrahedron $v$.](image)

With reference to Figure 2, the vector-valued Whitney function $w_{f_1}$ of degree 2, attached to the face $f_1$ (nodes $n_2$, $n_3$, $n_4$), is

$$w_{f_1} = 2 \left( w_{n_2} \nabla w_{n_3} \times \nabla w_{n_4} + w_{n_3} \nabla w_{n_4} \times \nabla w_{n_2} + w_{n_4} \nabla w_{n_2} \times \nabla w_{n_3} \right).$$

The basis functions of all other faces are found analogously. The cyclic order of the three nodes individuating a face $f_i$ is assumed to be that of the inner orientation of the face; a possible way to assign an inner orientation to a face $f_i$ is to choose an orientation to match the orientation of two bounding edges. For example (Figure 2), the orientation of the face $f_3$, matches that of the pair $e_2$ and $e_6$ with matching orientations.

The nodal function $w_{n_i}$ is an affine function that gets value 1 at node $n_i$, and value zero at all other nodes. At any point within a tetrahedron, one has

$$\sum_{i=1}^{4} w_{n_i} = 1. \quad (3)$$

By definition, the gradient of $w_{n_i}$ is a vector pointing towards the node $n_i$ and it is orthogonal to the face $f_i$ opposite to the node $n_i$. Therefore, it can be written as

$$\nabla w_{n_i} = \frac{u_i}{l_i},$$

where $l_i$ is the distance between the node $n_i$ and the face $f_i$, and $u_i$ is a unit vector normal to the face $f_i$. Let $\tilde{l}_i$ be a vector whose magnitude equals the area of the face $f_i$ and that is perpendicular to $f_i$ pointing in a way congruent (according to the screw rule) with the inner orientation of that face, then $u_i$ can be written as

$$u_i = -D_{v,i} \frac{f_i}{|\tilde{l}_i|},$$

where $D_{v,i}$ is the incidence number between the inner orientations of $v$ and face $f_i$. The inner orientation of $v$ is chosen
such that the counter-clockwise orientation of all its bounding faces is considered. Then (4) becomes
\[ \nabla w_{n_i} = -\frac{D_{e_{i,j}}}{3 \text{vol}(v)} f_i, \]
where \( \text{vol}(v) \) is the volume of the tetrahedron given by \( \text{vol}(v) = \frac{1}{3} | f_i | l_i / 3 \). The cross product between two area vectors \( f_i \times f_j \) is parallel to the common edge \( e_k \) that the faces \( f_i \) and \( f_j \) share, for example, \( f_3 \times f_4 \) is parallel to edge \( e_1 \) in Figure 3. Thanks to this property, the three cross products \( \nabla w_{n_i} \times \nabla w_{n_j} \), which appear in the definition of a face function such as (2), can be expressed in terms of edge vectors \( e_k \), whose amplitude is the length of the edge \( e_k \). For the sake of simplicity, let us focus on the orientations of the edges given in Figure 2. From elementary geometry and (6), we can derive that
\[ \nabla w_{n_i} \times \nabla w_{n_j} = \frac{D_{e_{i,j}}}{6 \text{vol}(v)} R_{i,k} e_k, \]
where \( R_{i,k} \) is the incidence number between the inner orientations of face \( f_i \) and edge \( e_k \). For example, see Figure 3, \( \nabla w_{n_3} \times \nabla w_{n_4} = \frac{1}{\text{vol}(v)} e_1 \), where \( e_1 \) is the edge vector associated with the edge \( e_1 \).

This is to say, that the vector field \( b \) is element-wise a constant. Of course, the field \( b \) complying with (10) can also be derived by expressing \( b \) as a linear combination of the constant vectors \( \text{rot } w_e = 2 \nabla w_{n_i} \times \nabla w_{n_j} \), where \( w_e \) is the edge vector-valued Whitney function of degree 1 associated with the edge \( e \). However, the way we showed this is more geometric.

Expressing the \( w_{f_i} \) in (9) according to (8) and using (10), we have
\[ 3 \text{vol}(v) b = b_1 [w_{n_2}(e_1 + e_5) + w_{n_3}(e_3 + e_6) + e_4(w_{n_1} + w_{n_4})] + b_2 [w_{n_1}(e_1 - e_4) + w_{n_5}(-e_2 - e_6) - e_5(w_{n_2} + w_{n_4})] + b_3 [w_{n_2}(-e_2 + e_5) + w_{n_1}(-e_3 + e_4) + e_6(w_{n_3} + w_{n_4})]. \]

Because for a face \( f_i \), the sum of its bounding edge vectors is null \( \sum_{j=1}^{3} R_{i,j} e_j = 0 \), then from (3) we obtain
\[ b = \frac{1}{3 \text{vol}(v)} (b_1 e_4 - b_2 e_5 + b_3 e_6). \]

Because also \( h = \nu b \) is uniform in \( v \), it follows that the magnetic voltages along the dual edge vectors \( \tilde{f}_i \) are simply
\[ h_i = h(P) \cdot \tilde{f}_i, \quad i = 1, \ldots, 4 \]
and independent of the choice of the point \( P \) in \( v \). Therefore, the elements of a possible constitutive matrix \( \nu \) for the considered mesh are
\[ \nu_{ij} = \nu \tilde{f}_i \cdot w_{f_j}(P), \quad i, j = 1, \ldots, 4. \]

Therefore \( w_{f_i} \) can be rewritten as a linear combination of the three edge vectors having in common the node \( n_i \) and resulting in
\[ w_{f_1} = \frac{1}{3 \text{vol}(v)} (w_{n_2} e_1 + w_{n_3} e_3 + w_{n_4} e_4), \]
\[ w_{f_2} = \frac{1}{3 \text{vol}(v)} (w_{n_1} e_1 - w_{n_3} e_2 - w_{n_4} e_5), \]
\[ w_{f_3} = \frac{1}{3 \text{vol}(v)} (-w_{n_1} e_3 - w_{n_2} e_2 + w_{n_4} e_6), \]
\[ w_{f_4} = \frac{1}{3 \text{vol}(v)} (+w_{n_1} e_4 + w_{n_2} e_5 + w_{n_3} e_6). \]

Then, thanks to (3), we can now write the basic property of these functions as \( w_{f_i}(P) \cdot \tilde{f}_j = \delta_{ij} \) when \( P \in f_j \) is satisfied.

Now, let us return to magnetostatics. If the magnetic flux density \( b \) is given in terms of Whitney facet elements, we have
\[ b = w_{f_1} b_1 + w_{f_2} b_2 + w_{f_3} b_3 + w_{f_4} b_4, \]
where the fluxes \( b_i \) relative to \( f_i \), with \( i = 1, \ldots, 4 \), comply also with the Gauss law, which, with the orientations of Figure 2, gives
\[ \sum_i D_{e_{i,j}} b_i = b_1 - b_2 + b_3 - b_4 = 0. \]
Moreover, considering the face functions \( w_{fi} \) in (8), evaluated at the barycenter, we have \( w_{ni}(\tilde{v}) = 1/4 \) with \( i = 1, \ldots, 4 \). From this and (15), it follows that \( \nu_{ij}(\tilde{v}) = \tilde{f}_i \cdot w_{fi}(\tilde{v}) = \tilde{f}_j \cdot w_{fi}(\tilde{v}) = \nu_{ji}(\tilde{v}) \).

A. Symmetry of the stiffness matrix

A possible way to satisfy the Gauss law (10) is to express the fluxes as

\[
b = R a,
\]

where \( a \) is the array of circulations of the magnetic vector potential along primal edges. For different choices of the point \( P \) we have then

\[
\nu(P) R = \nu(\tilde{v}) R.
\]

To solve (1) we can consider (16) together with Ampère’s law so that the resulting system becomes

\[
R^T \nu R a = j,
\]

where the system matrix \( R^T \nu R \) is symmetric due to (17). Moreover, in all cases the entries of the matrix \( R^T \nu R \) are precisely the same as those of the stiffness matrices one gets with the conventional finite element approaches. This shows that the finite element method does indeed have a geometric interpretation.

IV. CONSTITUTIVE MATRIX UNDER UNIFORMITY HYPOTHESIS

The key result obtained with (12), using the Whitney functions of degree 2, is that if the primal mesh is simplicial and \( D \) \( b = 0 \) holds, then the field \( b \) (or \( h = \nu b \)) within each tetrahedron \( v \) is uniform, that is, constant. Moreover, if the dual mesh is barycentric, and reluctivity is constant within each tetrahedron \( D \), and \( \nu \) holds and leads to the symmetry of the system matrix in (18). Therefore, let us revert to the initial hypothesis and as a starting point assume uniformity of the field \( b \) (and of the medium) within each \( v \); this, of course, implies \( D \) \( b = 0 \) in \( v \). We show here a possible way to compute directly a matrix \( \nu \) that satisfies (17).

In case of a single tetrahedron, three faces have always a common node. For each node \( n_i \) one may introduce a non-singular \( 3 \times 3 \) matrix \( S_i \), whose rows are components of the area vectors associated with the three faces sharing that node. For example, let us say node \( n_1 \) is a vertex of faces \( f_2, f_3, \) and \( f_4 \). Then, matrix \( S_1 \) is

\[
S_1 = \begin{bmatrix}
 f_{2x} & f_{2y} & f_{2z} \\
 f_{3x} & f_{3y} & f_{3z} \\
 f_{4x} & f_{4y} & f_{4z}
\end{bmatrix},
\]

where \( f_{ix}, f_{iy}, f_{iz} \) are the Cartesian components of the area vector \( f_i \). A uniform field \( b \) can now be derived from three fluxes as

\[
b = S_1^{-1} \begin{bmatrix}
 b_2 \\
 b_3 \\
 b_4
\end{bmatrix},
\]

and analogously for other matrices \( S_i, i = 1, \ldots, 4 \).

Now, to derive \( b \) from the array \( b \) of DoF’s, for each node we may introduce a \( 3 \times 4 \) matrix \( W_i \), which shares three columns with the inverse of \( S_i \), and, in addition, which has a column of zeroes in place of the linearly dependent flux. For example, the first column of \( W_1 \) related to the face opposite to node \( n_1 \) consists of zeroes, and the three other columns associated with the faces connected to node \( n_1 \) are those of the inverse of \( S_1 \).

Obviously, as the question is of linearly dependent fluxes, the choice of the node cannot affect \( b \), and therefore we have

\[
b = W_1 b = W_2 b = W_3 b = W_4 b.
\]

The very idea of the discrete constitutive law is to connect the fluxes \( b \) to the magnetomotive forces \( h \). We may now make different combinations of the \( W_i \) matrices to map the DoF-array \( b \) to \( h \). The following

\[
h = \nu \begin{bmatrix}
 \tilde{f}^{T} W_1 \\
 \tilde{f}^{T} W_2 \\
 \tilde{f}^{T} W_3 \\
 \tilde{f}^{T} W_4
\end{bmatrix} b = \nu' b
\]

leads to a non-singular matrix \( \nu' \) with a null diagonal and with three positive eigenvalues while the fourth is negative. The following combination

\[
h = \frac{1}{3} \nu \begin{bmatrix}
 \tilde{f}^{T} (W_2 + W_3 + W_4) \\
 \tilde{f}^{T} (W_1 + W_3 + W_4) \\
 \tilde{f}^{T} (W_1 + W_2 + W_4) \\
 \tilde{f}^{T} (W_1 + W_2 + W_3)
\end{bmatrix} b = \nu'' b
\]

leads to a non-singular \( \nu'' \) with positive eigenvalues. Finally, the combination

\[
h = \frac{1}{4} \nu \begin{bmatrix}
 \tilde{f}^{T} (W_1 + W_2 + W_3 + W_4) \\
 \tilde{f}^{T} (W_1 + W_2 + W_3 + W_4) \\
 \tilde{f}^{T} (W_1 + W_2 + W_3 + W_4) \\
 \tilde{f}^{T} (W_1 + W_2 + W_3 + W_4)
\end{bmatrix} b = \nu''' b
\]

leads to a singular \( \nu''' \). It can also be easily checked that \( \nu''' \) is symmetric and it coincides with \( \nu(\tilde{v}) \).

We can verify that

\[
\nu R = \nu' R = \nu'' R = \nu''' R,
\]

and thus, \( \nu, \nu', \nu'', \nu''' \) all lead to the same system matrix as in (18).

V. CONVERGENCE CONSIDERATIONS

Cast in differential forms, the reference continuous magnetostatic problem corresponding to (1) is

\[
db = 0, \quad h = \nu b, \quad dh = j,
\]

where \( \nu \) is now the Hodge operator between the 2–form \( b \) and the (twisted) 1–form \( h \), while \( d \) is the exterior derivative operator. (The metric-dependent counterparts of \( d \) are grad, curl, and div operators). Of course, \( d j = 0 \) is also satisfied, since \( dd = 0 \) and \( dh = j \) implies \( d dh = dj = 0 \). We denote with \( r_m \) the de Rham map, which sends a \( p–\)differential form such as \( b \) or \( h \) (twisted DF) to the corresponding arrays \( b \) and \( h \) of DoF, relative to the corresponding geometrical elements of \( \mathcal{K} \) and \( \bar{\mathcal{K}} \).
We discuss here the convergence of discrete problem (1) to differential problem (26) according to the results given in [11], [3], [2] by applying the de Rham map to problem (26) as follows

\[ D r_m b = 0, \quad r_m h = r_m \nu b, \quad R^T r_m h = j, \]

(27)

where we have used the following basic commutative property of the de Rham map:

\[ r_m dh = Dr_m b = 0, \quad r_m dh = R^T r_m h = r_m j = 0. \]

(28)

When we take the difference between the corresponding equations in discrete form (1) and discretized form (27) respectively, the following obtains for the Gauss law and Ampère’s law:

\[ D (b - r_m b) = 0, \quad R^T (h - r_m h) = 0, \]

(29)

whereas for constitutive equations

\[ h - r_m h = \nu b - r_m \nu b, \]

(30)

where the constitutive matrix \( \nu \) can be derived according to the uniformity hypothesis of the field within each cell \( v \) or equivalently using Whitney elements as in (14). If we rewrite (30) as

\[ h - r_m h = \nu b - r_m b - (r_m \nu b - \nu r_m b), \]

(31)

the term

\[ e_m = (r_m \nu - \nu r_m) b \]

(32)

represents consistency error.

If the actual field \( b \) were uniform mesh-wise (that is, uniform over each cell \( v \) of the mesh), then, as shown above, it could be described as

\[ b = \sum_f w_f b_f = p_m b \]

(33)

where \( p_m \) is the Whitney map, a tool which sends an array of DoF, such as \( b \), into a corresponding differential \( p \)-form, based on Whitney \( p \)-forms such as \( w_f \) with \( p=2 \). But by the definition of the \( \nu \) matrix in (14), such a field \( b \) makes the consistency error (32) vanish. Hence it follows from (31) and (29) that \( b = r_m b \) and \( h = r_m h \). In general, for a generic field \( b \), the convergence of \( p_m b \rightarrow b \) (and analogously of \( p_m h \rightarrow h \)) can yet be proved (see [11], [3], [2]), a fact which demonstrates that the norm of the consistency error tends to zero, as the grain \( \gamma_m \) of the mesh \( \mathcal{M} \), defined as the maximum diameter of the cells in both complexes \( \mathcal{K}, \mathcal{K}^\ast \), tends to zero. The key point is that the convergence could be proved even though a discrete Hodge \( \nu(P) \) in (14) was chosen non-symmetric.

VI. CONCLUSIONS

When implemented with Whitney forms, standard finite elements have noteworthy geometric properties, which become best visible when the subject is approached from the viewpoint of differential forms. In this paper, these geometric properties were demonstrated by reformulating the field problems according to a discrete algebraic approach such as the Cell method. The advantage of this geometric view-point is the complementary interpretation of discrete schemes designed to solve the magnetostatic field problem. This helps us to see what different numerical techniques have in common and emphasizes the geometrical nature of Maxwell’s equations.

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